Operational Quantum Gravity from Standard-Model Stress Response: Static Matching, Causal Tensor Dynamics, and Einstein-Equivalent Gravity

Online Resource 1
Supplementary Research Material
Technical Appendices A–H

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Appendix A. Closed-time-path normalization and constitutive inversion

Appendix A is the normalization and causality proof behind Secs. 2 and 3. The goal is to establish four points in one chain: why the source must be doubled on a closed-time-path (CTP) contour, where the factor $1/2$ in the stress response comes from, why the physical limit is taken only after differentiation, and why the inverse completed susceptibility is the causal compliance used in the matter-first description.

CTP is the standard Schwinger-Keldysh *in-in* formalism: it computes expectation values within one physical state rather than transition amplitudes between different asymptotic states. Once the bookkeeping is carried out consistently, the result is simple—the retarded stress Hessian and its inverse compliance are two orientations of the same constitutive relation. The construction follows Schwinger and Keldysh and modern nonequilibrium effective-action treatments by Calzetta and Hu.

Let $q_{+\mu\nu}$ and $q_{-\mu\nu}$ denote the forward and backward geometrical probe sources on the CTP contour. They couple to the same complete Standard-Model energy-momentum tensor on the two contour branches. With the symmetric rank-two source convention used in the body, the doubled source action is

$$S_{src}^{CTP}=\frac{1}{2}\int d^4x\left[q_{+\mu\nu}(x)\,T_+^{\mu\nu}(x)-q_{-\mu\nu}(x)\,T_-^{\mu\nu}(x)\right]. \tag{A1}$$

The factor $1/2$ is the conventional normalization for the contraction of two symmetric tensors. It is not a gravitational coupling and it is not adjustable later. Its only role is to fix what is meant by the source coordinate $q_{\mu\nu}$, and every response coefficient must be referred to the same convention.

For an initial density operator ρ_0, define the doubled generating functional by

$$Z[q_+,q_-]=Tr\left(U_{q_+}\rho_0U_{q_-}^{\dagger}\right),\,W[q_+,q_-]=-i\ln Z[q_+,q_-]. \tag{A2}$$

If the two sources are made equal, the forward and backward evolutions cancel by unitarity,

$$Z[q,q]=1\,,W[q,q]=0\,. \tag{A3}$$

Equation (A3) is why the contour cannot be collapsed before differentiation. The equality of the two branch sources is the *physical limit*, but the difference between the branches is the coordinate that generates physical expectation values and causal response.

Introduce the average and difference variables

$$q_r=\frac{q_++q_-}{2}\,,q_a=q_+-q_-\,, \tag{A4}$$

and, correspondingly,

$$T_r = \frac{T_+ + T_-}{2}, T_a = T_+ - T_- . \tag{A5}$$

The source bilinear then reorganizes exactly as

$$q_+ T_+ - q_- T_- = q_a T_r + q_r T_a , \tag{A6}$$

with tensor indices and spacetime integrations understood. Thus the two CTP branches are not two physical geometries. They are the doubled bookkeeping required to distinguish the applied physical perturbation from the variation used to read out its response.

The source-conjugate stress variables follow directly from Eqs. (A1)-(A6):

$$T_r^{\mu\nu}(x) = 2\frac{\delta W}{\delta q_{a\mu\nu}(x)}, T_a^{\mu\nu}(x) = 2\frac{\delta W}{\delta q_{r\mu\nu}(x)} . \tag{A7}$$

At the physical limit,

$$q_a = 0, T_a = 0, \tag{A8}$$

while q_r and T_r become the physical probe coordinate and physical expectation value. Equation (A7) displays the origin of the normalization used in Sec. 2: differentiating W with respect to the difference source produces one half of the physical stress because the source action itself contains one half.

For the following rank-four response equations, capital labels A, B denote condensed symmetric-tensor index pairs, including the associated spacetime arguments when no ambiguity can arise. The retarded susceptibility is the response of the physical stress to the physical average source,

$$\Pi_{AB}^R(x, y) \equiv \left.\frac{\delta T_{r,A}(x)}{\delta q_{r,B}(y)}\right|_{q_a=0} = 2\left.\frac{\delta^2 W}{\delta q_{a,A}(x)\delta q_{r,B}(y)}\right|_{q_a=0}, \tag{A9}$$

With the source convention of Eq. (A1), and with $\theta(t)$ denoting the Heaviside step function, the completed retarded response is

$$\Pi_{AB}^R(x, y) = -\frac{i}{2}\theta(x^0 - y^0)\langle[T_A(x), T_B(y)]\rangle + \Pi_{AB}^{R,contact}(x, y). \tag{A10}$$

The contact term in Eq. (A10) is the local contribution generated because the stress operator itself depends on the geometrical source. It is part of the same second functional variation as the separated-point commutator. Consequently the object to be inverted is the *completed* Hessian,

$$\Pi^R = \Pi_{conn}^R + \Pi_{contact}^R, \tag{A11}$$

not the connected correlator alone. This distinction is required by the stress Ward identities and becomes numerically important in the Higgs and dimensionally reduced matching calculations later in the paper.

For the conserved completed response, let p_μ denote the external four-momentum and let Π_{phys} denote the completed response after projection onto the physical tensor sector. The tangent-state Ward identity then takes the form

$$p_\mu \Pi_{phys}^{\mu\nu,\rho\sigma}(p)=0\,, \tag{A12}$$

after the contact terms required by the same metric variation have been included. Equation (A12) also explains why the full rank-four tensor Hessian cannot be inverted naively. Conservation and gauge redundancy produce null directions. Let P_{phys} denote the projector onto the non-gauge physical tensor subspace relevant to the response under consideration. The projected susceptibility is

$$\Pi_{phys}^{R}=P_{phys}\,\Pi^{R}\,P_{phys}\,, \tag{A13}$$

and its inverse is defined wherever this physical block is nonsingular,

$$D_{phys}^{R}\equiv\left(\Pi_{phys}^{R}\right)^{-1}. \tag{A14}$$

The poles of D^R are therefore zeros of the corresponding physical eigenvalues of Π^R. No inverse is assigned to a gauge or Ward-null direction.

The matter-first orientation is now obtained by a genuine doubled Legendre transform, performed before Eq. (A8) is imposed. This is the usual CTP effective-action construction applied to the stress/source pair rather than an additional microscopic dynamics. Define

$$\Gamma\left[T_r,T_a\right]=W\left[q_r,q_a\right]-\frac{1}{2}\int d^4x\left(q_{a\mu\nu}T_r^{\mu\nu}+q_{r\mu\nu}T_a^{\mu\nu}\right). \tag{A15}$$

Using Eq. (A7), variation gives the exact conjugacy relations

$$\frac{\delta\Gamma}{\delta T_r^{\mu\nu}(x)}=-\frac{1}{2}q_{a\mu\nu}(x)\,,\;\frac{\delta\Gamma}{\delta T_a^{\mu\nu}(x)}=-\frac{1}{2}q_{r\mu\nu}(x). \tag{A16}$$

Only after these derivatives have been defined is the physical limit taken. On that branch the second relation in Eq. (A16) reads

$$q_{r,A}(x)=-2\frac{\delta\Gamma}{\delta T_{a,A}(x)}. \tag{A17}$$

Differentiating Eq. (A17) with respect to the physical stress and using Eqs. (A9), (A13) and (A14) gives

$$\frac{\delta q_r}{\delta T_r}=D_{phys}^{R}=\left(\Pi_{phys}^{R}\right)^{-1}, \tag{A18}$$

or, equivalently,

$$\left.\frac{\delta^2\Gamma}{\delta T_a\,\delta T_r}\right|_{phys}=-\frac{1}{2}D_{phys}^{R}. \tag{A19}$$

Equations (A18) and (A19) are the technical core of the matter-first inversion. They show directly that the mixed curvature of the stress-space functional is the inverse of the same retarded Standard-Model susceptibility that was obtained by differentiating the probe functional. Nothing has been added to the microscopic theory. The Legendre transform changes which member of a conjugate response pair is used as the independent variable.

The corresponding linear constitutive equations can therefore be written in either orientation,

$$\delta T_r = \Pi^R_{phys}\,\delta q_r\,,\,\delta q_r = D^R_{phys}\,\delta T_r\,. \tag{A20}$$

This equality is the precise field-theory form of the elementary constitutive analogy used in Sec. 3. Writing $F = kx$ or $x = F/k$ does not create a new spring; writing polarization as a response to electric field or the conjugate field as a response to specified polarization does not change the dielectric. In the same way, Eq. (A20) expresses one completed Standard-Model tensor response in the orientation needed for a matter-first description. Stress-space and inverse-correlator constructions provide independent field-theoretic precedent for this operation [3].

The physical matter perturbation may now be made explicit. Let the Standard-Model fields and state variables be denoted collectively by Φ. A physical change of state produces

$$\delta T_{r,A}(x) = \int d^4 y\,\frac{\delta T_{r,A}(x)}{\delta \Phi_I(y)}\,\delta \Phi_I(y)\,, \tag{A21}$$

and therefore

$$\delta q_{r,A}(x) = \int d^4 y\, D^R_{phys,AB}(x,y)\,\delta T_{r,B}(y)\,. \tag{A22}$$

Equation (A22) is the causal ordering used throughout OQG: the matter state changes, its complete stress changes, and the inverse completed susceptibility assigns the corresponding operational tensor response. No equation obtained by varying the probe source to zero is imposed on ordinary Standard-Model configurations.

The same result can be written as a reduced matter constitutive functional by composing $\Gamma[T_r, T_a]$ with the stress doublet generated by the Standard-Model fields. To quadratic order, the retarded cross term is

$$\Gamma^{(2)}_{red,ret} = -\frac{1}{2}\int d^4 x\, d^4 y\,\delta T_{a,A}(x)\, D^R_{phys,AB}(x,y)\,\delta T_{r,B}(y) + \cdots\,, \tag{A23}$$

where the ellipsis denotes the symmetric/noise and local pieces required by the complete CTP functional. Varying Eq. (A23) reproduces Eq. (A22). Thus the matter-only description and the source-response description are not competing constructions; they are Legendre-dual representations of the same causal Hessian.

Three consistency checks follow immediately. First, the reverse CTP branch has no interpretation as backward physical time: it disappears from the physical limit after doing the causal differentiations. Second, the factor $1/2$ in the retarded commutator is fixed by Eq. (A1), so changing that normalization would rescale both Π^R and its inverse and would also spoil the later Kubo-to-stiffness normalization. Third, Ward/contact completion must precede the physical inversion; otherwise one attempts to invert an object that does not possess the conserved tensor structure of the original Standard-Model functional.

The result established here is therefore stronger than a notation change but narrower than the introduction of a new microscopic field. The CTP construction fixes the causal normalization, the doubled Legendre transform is well defined before the physical limit, and the projected stress-space Hessian is exactly the inverse completed retarded susceptibility. This proves the constitutive step used in Sec. 3: matter stress can be treated as the physical input and the source-conjugate operational tensor as the response coordinate without changing the underlying Standard-Model dynamics. The fixed-measure/WTDiff interpretation is deliberately applied only after this reduction; it is not assumed anywhere in the normalization or inversion derived above.

Appendix B. Transverse-traceless (TT) projection, physical spectral weights, and the free-vacuum positivity bound

Appendix B isolates the physical spin-two content of the Standard-Model stress response. The transverse-traceless (TT) projector removes longitudinal and trace pieces, leaving the shear-like tensor sector that carries the two helicities of a massless spin-two response in the infrared. This lets three questions be answered cleanly: what the physical scalar, Weyl, and vector spectral weights are; why the resolved spin-two spectral density is positive; and which free-field coefficient enters the coherent causal kernel.

The result is $N_T = 283$ for the minimal Standard Model. This number is a property of the *resolved physical TT spectrum*. It is neither a fitted gravitational normalization nor the same object as the static constitutive stiffness calculated in Sec. 4 and Appendix C.

Let $\eta_{\mu\nu}$ denote the local tangent-space Lorentz metric. For a four-momentum p^μ away from the null point, define the transverse tensor

$$\theta_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{B1}$$

In four spacetime dimensions the spin-two projector acting on a symmetric rank-two tensor is

$$P^{(2)}_{\mu\nu,\rho\sigma}(p) = \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}, \tag{B2}$$

while the transverse scalar projector is

$$P^{(0)}_{\mu\nu,\rho\sigma}(p) = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}. \tag{B3}$$

The spin-two projector is transverse and traceless,

$$p^\mu P^{(2)}_{\mu\nu,\rho\sigma} = 0, \eta^{\mu\nu} P^{(2)}_{\mu\nu,\rho\sigma} = 0, \tag{B4}$$

and it is idempotent on the physical tensor subspace. These properties are the algebraic reason the TT sector can be discussed independently once the completed stress Hessian has satisfied the Ward identity derived in Appendix A.

For a conserved stress tensor, the two-point function admits a Källén-Lehmann decomposition into independent spin-two and scalar spectral pieces [15]. Schematically,

$$\rho_{\mu\nu,\rho\sigma}(p) = \rho_2(p^2) P^{(2)}_{\mu\nu,\rho\sigma}(p) + \rho_0(p^2) P^{(0)}_{\mu\nu,\rho\sigma}(p), \tag{B5}$$

up to convention-dependent overall normalization and contact polynomials. The physical spin-two density is nonnegative in a unitary theory,

$$\rho_2(s) \geq 0. \tag{B6}$$

The reason is direct. After TT projection, the spectral density is a sum over physical intermediate states of squared stress-tensor matrix elements,

$$\rho_2(s) \propto \sum_n (2\pi)\,\delta\left(s - m_n^2\right)\left|\langle 0|T^{TT}|n\rangle\right|^2 . \tag{B7}$$

Equation (B7) is the positivity statement used throughout the coherent-channel analysis. It applies to the resolved absorptive spin-two spectral response. Local subtraction terms, metric seagulls and trace-sector contacts do not become additional positive spectral states merely because they occur in the same completed Hessian.

For free four-dimensional fields the normalized spin-two stress coefficients are

$$C_T^{(s)} = \frac{1}{120},\, C_T^{(W)} = \frac{1}{40},\, C_T^{(V)} = \frac{1}{10}, \tag{B8}$$

for one real scalar, one Weyl fermion and one gauge vector, respectively [15]. Dividing by the scalar unit gives the particularly transparent ratio

$$C_T^{(s)} : C_T^{(W)} : C_T^{(V)} = 1:3:12 . \tag{B9}$$

This ratio is not a count of helicities. It is the relative strength with which each free field contributes to the physical spin-two stress correlator in one common normalization. The gauge-vector coefficient already refers to the physical, Ward-completed tensor response rather than to a naive count of gauge-potential components.

The minimal Standard Model contains four real Higgs components, 45 Weyl fermions and 12 gauge vectors. Therefore its free coherent TT weight is

$$\begin{aligned} N_T &= 4(1) + 45(3) + 12(12) \\ &= 4 + 135 + 144 \\ &= 283 . \end{aligned} \tag{B10}$$

Equivalently, in the conventional stress-central-function normalization,

$$C_{T,SM} = 4\left(\frac{1}{120}\right) + 45\left(\frac{1}{40}\right) + 12\left(\frac{1}{10}\right) = \frac{283}{120} . \tag{B11}$$

Several useful conclusions follow immediately from Eq. (B10). First, the gauge sector carries slightly more than half of the free coherent TT weight, the Weyl sector carries slightly less than half, and the four Higgs components contribute only a small fraction. Second, all three contributions enter with the same positive spectral sign. Third, 283 characterizes a nonlocal coherent matter moment; it does not by itself determine Newton's constant or the static coefficient $A_{2,E}$.

This last distinction is important because a superficially similar one-loop heat-kernel expression has sometimes been written with the combination

$$N_0 + N_{1/2} - 4N_1 . \tag{B12}$$

That quantity is not the physical TT spectral count. Its negative vector term already shows that it cannot be a sum of the nonnegative matrix-element weights in Eq. (B7). It is a local proper-time/heat-kernel coefficient in which gauge, ghost, seagull, subtraction and contact information are combined in the coefficient of a local curvature operator [6]. Such a coefficient is a legitimate object of the renormalized effective action, but it answers a different question from the resolved TT spectral density. The distinction is one of the principal reasons the present calculation keeps coherent spectral response, local Ward/contact response and static thermodynamic stiffness separate until the completed observable is formed.

The same TT projection also fixes the role of the Higgs curvature improvement. For one scalar,

$$\Delta T_{\mu\nu}^{(\xi)} = \xi\left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right)\phi^2 . \tag{B13}$$

In momentum space this contribution is proportional to

$$\eta_{\mu\nu} p^2 - p_\mu p_\nu . \tag{B14}$$

Applying Eq. (B2),

$$P^{(2)\,\mu\nu,\rho\sigma}\left(\eta_{\rho\sigma} p^2 - p_\rho p_\sigma\right) = 0 , \tag{B15}$$

because the metric term is removed by tracelessness and the momentum term by transversality. Hence

$$P^{(2)} \Delta T^{(\xi)} = 0 . \tag{B16}$$

The resolved physical TT spectral weight of the Higgs sector is therefore independent of ξ_H. This does not make the Higgs-curvature operator irrelevant. Curved-space Standard-Model renormalization requires that operator [19], and its second metric variation contributes to local/contact and static variational response. Section 4 and Appendix C include precisely that effect. Equation (B16) states only that ξ_H cannot be used to manufacture additional propagating TT spectral states or to alter the $1:3:12$ weight.

A final positivity result is retained because it sharpened the physical interpretation of the coherent channel. Consider a unitary Lorentz-invariant single TT eigenchannel with twice-subtracted inverse response

$$K(t) = At + Bt^2 + t^2 \int_0^\infty ds \, \frac{\rho_2(s)}{s^2 (s - t)} , \rho_2(s) \geq 0 . \tag{B17}$$

Suppose one asks whether this isolated positive channel can possess a nondegenerate double zero at a negative invariant point $t = -x$, with $x > 0$. The two conditions

$$K(-x) = 0 , K_t(-x) = 0 \tag{B18}$$

give

$$B = -\int_0^\infty ds \, \frac{\rho_2(s)}{s (s + x)^2} < 0 , \tag{B19}$$

and

$$A = x^2 \int_0^\infty d\,s \frac{\rho_2(s)}{s^2 (s+x)^2}. \tag{B20}$$

Since

$$0 < \frac{x^2}{(s+x)^2} < 1$$

for every $s > 0$,

$$0 < A < A_{resolved}, A_{resolved} \equiv \int_0^\infty d\,s \frac{\rho_2(s)}{s^2}. \tag{B21}$$

Equation (B21) is a useful theorem about the restricted free-vacuum construction: if one tries to make an isolated positive resolved TT channel simultaneously supply both an enhanced local stiffness and an imaginary-axis double zero, the double-zero condition constrains the local coefficient to lie *below* the positive resolved moment. The earlier sharp-endpoint realization instead required the opposite ordering. That overly restrictive identification is therefore excluded without altering the physical TT count itself.

The significance of this bound is constructive. It forced the calculation to separate three objects that had previously been too easy to identify with one another: the positive coherent spectral moment, the state-dependent static variational stiffness, and any extreme critical/self-loading condition. A finite Kubo-Martin-Schwinger (KMS)-state Kubo coefficient is not constrained by Eq. (B21) in the same way because it is a thermodynamic derivative of a state-dependent Hessian rather than a positive Lorentz-invariant absorptive moment; its scalar contribution can even be negative. Likewise, the ordinary causal branch derived later is a simple underdamped resonance rather than a free-vacuum double zero. The present manuscript therefore retains the positive $N_T = 283$ result while no longer requiring ordinary weak-coupling matter to satisfy an extreme critical boundary condition.

Appendix B thus establishes the spectral side of the two-point program. The completed Standard-Model stress response contains a positive physical spin-two sector; its free field weights are fixed in the ratio $1:3:12$; the minimal field content gives the coherent coefficient $N_T = 283$; the Higgs improvement operator is exactly TT-null in this resolved sector; and positivity itself prevents a local heat-kernel coefficient or an arbitrary critical closure from being reinterpreted as additional physical spectral weight. Appendix C turns to the complementary static moment of the same matter response and derives the Kubo stiffness, including the Higgs variational contact that is absent from the resolved TT spectrum.

Appendix C. Euclidean Kubo stiffness, Higgs variational contact, and leading soft Standard-Model response

Appendix C supplies the static calculation behind Sec. 4. The observable is a hydrostatic Kubo coefficient: the second derivative of the equilibrium stress response with respect to a slowly varying TT shear. Physically, it measures how strongly the matter state resists a long-wavelength tensor deformation. Because it is a derivative of the *completed metric Hessian*, it includes local metric-variation contacts as well as separated-point correlators.

Three results are derived here. A finite equilibrium state generates a two-derivative TT stiffness without a physical ultraviolet cutoff; the free minimal Standard Model produces the characteristic coefficient 133, with the Higgs-curvature dependence shown explicitly; and resummation of the bosonic zero Matsubara modes gives the first nonanalytic interaction correction, $O(g)$. This also explains why the Higgs improvement can affect the static variational Hessian even though it contributes no additional resolved TT spectral weight in Appendix B.

Take a stationary isotropic Kubo-Martin-Schwinger (KMS) state and introduce a static off-diagonal shear source propagating along the z direction,

$$h_{xy}(z)=h_k e^{ikz}+h_{-k} e^{-ikz}. \tag{C1}$$

Because the only nonzero perturbation is h_{xy} and the external momentum is parallel to z, the source is transverse and traceless. In the conventional Euclidean metric-variation normalization, define the hydrostatic shear coefficient using the Kubo relation of Moore and Sohrabi; the distinction between the separated-point operator correlator and the full metric-variational Hessian follows the formulation of Kovtun and Shukla:

$$\kappa(T)=\frac{\partial^2}{\partial k^2} G_E^{xy,xy}(0,k)\bigg|_{k=0}. \tag{C2}$$

Equivalently,

$$G_E^{xy,xy}(0,k)=G_E^{xy,xy}(0,0)+\frac{\kappa(T)}{2}k^2+O(k^4). \tag{C3}$$

The completed Hessian in Eq. (C2) contains both the separated-point correlator and the local metric contact generated by the same microscopic action. Momentum-independent seagulls affect $G_E(0,0)$ but disappear from the k^2 derivative; a contact containing two metric derivatives, such as the scalar curvature-improvement term, survives and must be retained.

The OQG source convention carries the symmetric factor $1/2$ derived in Appendix A. With

$$\Pi_{TT,E}=-\frac{1}{2}G_E^{xy,xy}, \tag{C4}$$

the two-derivative constitutive coefficient is therefore

$$A_{2,E}(T)=-\left.\frac{\partial \Pi_{TT,E}}{\partial k^2}\right|_0=\frac{\kappa(T)}{4}. \tag{C5}$$

Equation (C5) is the normalization used throughout the manuscript. It does not refer to a measured gravitational constant, a sharp spectral endpoint, or an external ultraviolet ruler.

The origin of the state-dependent k^2 term can already be seen in the free scalar calculation. For one real massless scalar with operator stress $T_{xy}=\partial_x\phi\,\partial_y\phi$, the thermal two-point bubble is

$$G^{op}_{E,s}(0,k)=2(k_B T)\sum_n\int_p \frac{p_x^2 p_y^2}{P^2(P+K)^2}, \tag{C6}$$

where

$$P=(\omega_n, p), K=(0,0,0,k), \omega_n=2\pi n k_B T.$$

Feynman parameterization and the shift of the loop momentum give

$$G^{op}_{E,s}(0,k)=\frac{k_B T}{12\pi}\int_0^1 dx\sum_n\left[\omega_n^2+x(1-x)k^2\right]^{3/2}, \tag{C7}$$

with the zero-temperature/local piece understood in the same renormalized completed response. The angular reduction uses

$$\langle p_x^2 p_y^2\rangle=\frac{p^4}{15}, \int_p \frac{p_x^2 p_y^2}{\left(p^2+M^2\right)^2}=\frac{M^3}{24\pi}. \tag{C8}$$

Differentiating Eq. (C7) at $k=0$ and performing the thermal sum gives the operator contribution

$$\frac{\kappa_s^{op}}{(k_B T)^2}=-\frac{1}{72}. \tag{C9}$$

This negative hydrostatic coefficient does not conflict with the positive spin-two spectral density of Appendix B. The two quantities are different moments of the response: Eq. (C9) is a static thermodynamic derivative, whereas Appendix B concerns a resolved absorptive spectral measure.

For a scalar with curvature improvement

$$S_\xi=-\frac{1}{2}\int d^4x\sqrt{-g}\,\xi R\phi^2, \tag{C10}$$

the first flat-space stress improvement is TT-null, as shown in Eq. (B16). The *second* metric variation is different. It generates the local curvature contact required by the variational Hessian. In the static shear channel its contribution can be written

$$\kappa_{s,contact}=\xi\langle\phi^2\rangle_T. \tag{C11}$$

For one free massless real scalar,

$$\langle \phi^2 \rangle_T = \frac{(k_B T)^2}{12}, \tag{C12}$$

so the complete variational scalar coefficient is

$$\frac{\kappa_s(\xi)}{(k_B T)^2} = -\frac{1}{72} + \frac{\xi}{12} = -\frac{1-6\xi}{72}. \tag{C13}$$

This equation is the clean separation needed for the Higgs sector: the operator/resolved TT piece is independent of ξ, while the local metric Hessian is not.

For one Weyl fermion and one gauge vector, the corresponding completed free Kubo coefficients are

$$\frac{\kappa_W}{(k_B T)^2} = \frac{1}{144}, \frac{\kappa_V}{(k_B T)^2} = \frac{1}{18}. \tag{C14}$$

The Weyl result follows from one half of the Dirac Belinfante loop. For the gauge field, the field-strength stress vertex is transverse on the incoming gauge legs, and the Ward-completed result carries the two physical vector polarizations. Independent finite-k calculations reproduce both slopes in Eq. (C14), providing a check on the normalization used here.

The minimal Standard Model contains four real Higgs components, 45 Weyl fermions and 12 gauge vectors. Applying Eq. (C13) with the common Higgs curvature coupling ξ_H gives

$$\begin{aligned}\frac{\kappa_{SM}^{(0)}}{(k_B T)^2} &= 4\left[-\frac{1-6\xi_H}{72}\right] + 45\left(\frac{1}{144}\right) + 12\left(\frac{1}{18}\right) \\ &= \frac{133+48\xi_H}{144}.\end{aligned} \tag{C15}$$

Using Eq. (C5),

$$\frac{A_{2,E}^{(0)}}{(k_B T)^2} = \frac{133+48\xi_H}{576}. \tag{C16}$$

At minimal curvature coupling,

$$\frac{A_{2,E}^{(0)}}{(k_B T)^2} = \frac{133}{576} = 0.230902777778. \tag{C17}$$

The integer 133 is therefore the minimal-coupling member of the free **variational** Standard-Model family. It should not be identified with the coherent spectral coefficient 283. The former is a static metric derivative; the latter is the positive resolved spin-two weight.

The first interaction correction is controlled by the bosonic zero Matsubara modes. The zero mode of Eq. (C7) is nonanalytic when left massless,

$$G^{op}_{E,s,0}(0,k)=\frac{k_B T|k|^3}{512}. \tag{C18}$$

Screening in the physical thermal state replaces the massless zero mode by a resummed three-dimensional mode of mass m,

$$G^{op}_{E,s,0}(0,k;m)=\frac{k_B T}{12\pi}\int_0^1 dx\left[m^2+x(1-x)k^2\right]^{3/2}. \tag{C19}$$

Its low-momentum derivative gives

$$\delta\kappa^{op}_{s,0}=\frac{(k_B T)m}{24\pi}. \tag{C20}$$

The curvature contact has its own soft correction. The cancellation pattern at conformal scalar coupling is also consistent with the interacting thermodynamic-transport analysis of Weiner and Romatschke, where the corresponding curvature susceptibility is generated only once interactions provide a thermal scale. The zero-mode condensate changes by

$$\Delta\langle\phi^2\rangle_0=-\frac{(k_B T)m}{4\pi}, \tag{C21}$$

and hence the Ward-completed result for one resummed scalar zero mode is

$$\delta\kappa^{(g)}_{s,0}=\frac{(k_B T)m}{24\pi}(1-6\xi). \tag{C22}$$

Equation (C22) supplies a useful internal check: for a conformally coupled scalar, $\xi=1/6$, the free plus contact zero-mode contribution cancels at this order. Temporal gauge scalars use $\xi=0$. Fermions have no zero Matsubara mode and therefore do not contribute to the $O(g)$ ring term.

For reproducibility, the notation used in the running and screening benchmark is fixed here before the numerical table. The four-dimensional modified-minimal-subtraction scale is $\acute{\mu}$. The gauge couplings g_Y, g_2, g_3 are the hypercharge, weak-isospin, and strong couplings; y_t, y_b, y_τ are the top, bottom, and tau Yukawa couplings; λ is the Higgs quartic coupling; and $Y_2\equiv 3\left(y_t^2+y_b^2\right)+y_\tau^2$ is the retained Yukawa invariant. The screening masses $m_{D3}, m_{D2}, m_{DY}, m_H$ denote the color-electric, weak-electric, hypercharge-electric, and Higgs screening masses. The high-temperature matching organization follows the Standard-Model dimensional-reduction framework of Kajantie, Laine, Rummukainen and Shaposhnikov. The central high-state running prescription is

$$\acute{\mu}=2\pi k_B T, \tag{C23}$$

with the perturbative scale varied over

$$\pi k_B T\le\acute{\mu}\le 4\pi k_B T. \tag{C24}$$

This is a renormalization prescription for evaluating the Standard-Model couplings, not a physical ultraviolet cutoff. At the central point the running inputs are summarized in Table C1.

Table C1. Running Standard-Model couplings at the central high-state renormalization point.

Running quantity	Central value at $\acute{\mu}=2\pi k_B T$
g_Y	0.477570988239
g_2	0.502460397148
g_3	0.486635235305
y_t	0.401965124496
y_b	0.005647150306
y_τ	0.009453193549
λ	-0.033586864198
Y_2	0.484912917721

The corresponding leading screening-mass ratios are

$$\frac{m_{D3}}{k_B T}=0.688206\,,\;\frac{m_{D2}}{k_B T}=0.680335\,,\;\frac{m_{DY}}{k_B T}=0.646634\,,\;\frac{m_H}{k_B T}=0.291904\,. \tag{C25}$$

There are eight color-electric modes, three weak-electric modes, one hypercharge-electric mode and four real Higgs modes. The multiplicity-weighted mass sum at $\xi_H=0$ is therefore

$$8\,m_{D3}+3\,m_{D2}+m_{DY}+4\,m_H=9.360904\,k_B T\,. \tag{C26}$$

Using Eq. (C22) and then dividing by four according to Eq. (C5), the complete soft correction is

$$\frac{\delta A_{2,E}^{(g)}}{(k_B T)^2}=\frac{8\,m_{D3}+3\,m_{D2}+m_{DY}+4\,m_H(1-6\,\xi_H)}{96\,\pi\,k_B T}\,. \tag{C27}$$

At the central running point this gives

$$\frac{\delta A_{2,E}^{(g)}}{(k_B T)^2}=0.031038211243-0.0232268807\,\xi_H\,, \tag{C28}$$

so Eqs. (C16) and (C28) combine to

$$\frac{A_{2,E}^{(0+g)}}{(k_B T)^2}=0.261940989021+0.0601064526\,\xi_H\,. \tag{C29}$$

For $\xi_H=0$, the soft resummation increases the free value by approximately

$$13.44\,\%\,. \tag{C30}$$

The leading result is also numerically stable under the stated factor-of-four scale variation. At $\xi_H = 0$, the result is summarized in Table C2.

Table C2. Renormalization-scale variation of the leading static stiffness at minimal Higgs-curvature coupling.

Renormalization point	$A_{2,E}^{(0+g)} / (k_B T)^2$
$\pi k_B T$	0.262102908294
$2\pi k_B T$	0.261940989021
$4\pi k_B T$	0.261783383062

Across this interval the variation is about 0.12%. This small residual sensitivity is a consistency check on the leading thermal benchmark; it is not used to replace the explicit $O(g^2)$ matching.

The physical accomplishment of the Kubo calculation is therefore independent of the later two-loop bookkeeping. A specified Standard-Model state supplies a finite two-derivative TT stiffness directly from its equilibrium response. The free coefficient is calculable, the local Higgs-curvature dependence is identified unambiguously as a metric-variational contact rather than additional spectral weight, and the leading infrared-sensitive interaction correction is finite after zero-mode resummation. The earlier sharp-boundary value $\xi_H = 1/24 - \pi$ is consequently not retained as a resolved-TT prediction; ξ_H remains a renormalized Standard-Model input in the static Hessian. Appendices D, E and F build on the normalization fixed here to complete the hard $O(g^2)$, electric/Higgs/local-curvature, and magnetostatic matching without altering the Kubo definition itself.

Appendix D. Strict-hard Standard-Model $O(g^2)$ TT operator reduction

Appendix D calculates the *strict-hard* part of the $O(g^2)$ static stiffness: the contribution from thermal loop momenta of order $\pi k_B T$, after the bosonic zero modes have been reserved for the lower-energy three-dimensional theory of Appendix E. The target is the coefficient of the external k_z^2 term in the complete two-loop TT operator correlator.

This is the most algebraic part of the static calculation, so a few terms are useful up front. A **master integral** is a standard loop integral to which many diagrams are reduced; **integration by parts (IBP)** identities relate those masters; and an **evanescent term** is proportional to the dimensional-regularization parameter and can survive when multiplied by a $1/\epsilon$ ultraviolet pole. The value of the calculation is not only its numerical hard coefficient: it also provides gauge-independent sector organization, exact cancellations, mixed Bose/Fermi IBP checks, and the residual ultraviolet ledger that must be closed by the variational contact and three-dimensional matching.

The starting Euclidean interaction term is the complete general-momentum Standard-Model TT operator result of Ghiglieri, Jackson, Laine and Zhu [12]. Let $G_{12;12}^{E,(2)}(K)$ denote their two-loop Euclidean $12,12$ shear-channel operator correlator at external Euclidean four-momentum K. The field labels s, f, g denote scalar, fermion, and gauge sectors. A master combination $\Phi_{x(y)}$ denotes the contribution with sector x in the indicated operator/topology class and sector y in the coupled loop family, while the vertical-bar notation $\Phi_{x|y}$ denotes the corresponding mixed/interference family in the published decomposition. The couplings $\lambda, g_2, g_3, g_Y, y_t$ are respectively the Higgs quartic, weak, strong, hypercharge, and top-Yukawa couplings. In four dimensions the result can be organized into five gauge-independent sectors,

$$
\begin{aligned}
G_{12;12}^{E,(2)}(K) = \frac{1}{4}\{ & \lambda \Phi_{s(s)} + \left(3 g_2^2 + 12 g_3^2\right) \Phi_{g(g)} \\
& + 3 y_t^2 \left(\Phi_{s(f)} + \Phi_{f(s)} + \Phi_{s|f}\right) \\
& + \left(g_Y^2 + 3 g_2^2\right)\left(\Phi_{s(g)} + \Phi_{g(s)} + \Phi_{s|g}\right) \\
& + \left(5 g_Y^2 + 9 g_2^2 + 24 g_3^2\right)\left(\Phi_{f(g)} + \Phi_{g(f)} + \Phi_{f|g}\right)\}.
\end{aligned}
\tag{D1}
$$

The five combinations will be denoted by the strict-hard slopes

$$
H_\lambda, H_{\text{gg}}, H_y, H_{sg}, H_{fg},
\tag{D2}
$$

where, for example,

$$
H_{fg} \equiv \frac{1}{\left(k_B T\right)^2} \partial_{k_z}^2 \left(\Phi_{f(g)} + \Phi_{g(f)} + \Phi_{f|g}\right)^{hard}\Big|_{k_z=0}.
\tag{D3}
$$

All five slopes are defined with the same external Euclidean momentum,

$$
K = (0, 0, 0, k),
\tag{D4}
$$

and with

$$D = 4 - 2\epsilon\,, d = 3 - 2\epsilon \tag{D5}$$

kept until the integration-by-parts reduction and Laurent expansion are complete. This dimensional discipline is essential: one of the largest hard contributions vanishes algebraically at exactly $d = 3$ and survives only because its $O(\epsilon)$ coefficient multiplies a $1/\epsilon$ tadpole pole.

For Eq. (D6), $\sum_{P,Q}$ denotes the standard dimensionally regulated double thermal sum-integral, schematically $(k_B T)^2 \sum_{p_n, q_n} \int_p \int_q$, with the spatial integrals carrying the usual modified-minimal-subtraction measure. The symbols $P_T, Q_T, (Q - P)_T$ denote the corresponding Matsubara-frequency components $p_n, q_n, q_n - p_n$. In the published master basis, undecorated, barred, tilded, and hatted families distinguish the Bose/Fermi assignments of the loop frequencies; each Φ sector in Eq. (D1) inherits its prescribed assignment. The thermal IBP/factorization basis used below follows the two-loop sum-integral reduction developed by Schröder and used in the Ghiglieri-Jackson-Laine-Zhu master representation. The published two-loop masters have the generic form

$$I_{abcde}^{fgh} = \sum_{P,Q} \frac{(P_T)^f (Q_T)^g \left[(Q-P)_T\right]^h (K^2)^y}{(P^2)^a (Q^2)^b \left[(Q-P)^2\right]^c \left[(K-P)^2\right]^d \left[(K-Q)^2\right]^e}\,, \tag{D6}$$

with

$$y = a + b + c + d + e - f - g - h - 2\,. \tag{D7}$$

Equation (D7) makes the static reduction almost mechanical. A term with $y \geq 2$ begins at k^4 and cannot contribute to the stiffness. A term with $y = 1$ already carries one explicit factor of $K^2 = k^2$, so the remaining vacuum integral is evaluated at $K = 0$. A $y = 0$ term contributes only if an external propagator remains; if both external propagator powers vanish, the term may retain the direction used to define the TT projector but is independent of $|k|$, and its second derivative is zero.

For the genuine $y = 0$ terms the required Taylor operator is

$$\partial_k^2 \frac{1}{\left[(P-K)^2\right]^d \left[(Q-K)^2\right]^e}\Bigg|_0 = \frac{1}{(P^2)^d (Q^2)^e}\Big[-\frac{2d}{P^2} - \frac{2e}{Q^2} + \frac{4d(d+1)p_z^2}{(P^2)^2} + \frac{4e(e+1)q_z^2}{(Q^2)^2} + \frac{8de\,p_z q_z}{P^2 Q^2}\Big]. \tag{D8}$$

All odd spatial tensors vanish after angular integration. The surviving transverse numerators are reduced with exact d-dimensional isotropic averages before any $d \to 3$ limit is taken. Bosonic zero-mode pieces that become scaleless in the strict massless expansion are absent from the hard contribution; their resummed physical counterparts are precisely the electric/Higgs and magnetostatic sectors treated later.

The Higgs self-coupling sector is the simplest example. The relevant master factorizes into a one-loop tadpole times a one-loop static bubble, and the nonzero-mode Matsubara sum gives

$$H_\lambda = \frac{1}{12\pi^2\epsilon} + \frac{-48\zeta'(-1) + 8\ln(\acute{\mu}/4\pi k_B T) - 3 + 4\gamma_E}{24\pi^2} + O(\epsilon). \tag{D9}$$

At $\acute{\mu} = 2\pi k_B T$, minimal subtraction gives

$$H_\lambda^{\overline{MS}} = 0.00719338234\,. \tag{D10}$$

With the Standard-Model weight $C_\lambda = \lambda/4 = -0.008396725$, the corresponding OQG stiffness contribution is

$$\frac{\delta A_{\lambda,hard}}{(k_B T)^2} = -0.000015100213339\,. \tag{D11}$$

The gauge-gauge sector is structurally more revealing. Static power counting reduces the published expression to fifteen $y=1$ zero-momentum structures and seven genuine $y=0$ Taylor structures. Before integration, one pair cancels exactly by permutation symmetry,

$$4 I_{11101}^{100} - 4 I_{21100}^{010} = 0\,, \tag{D12}$$

and a second four-master combination collapses to a single tadpole product,

$$-12 I_{12001}^{010} + 4 I_{121-11}^{100} - 4 I_{12001}^{100} + 8 I_{12000}^{000} = 2(5-d) I_1 I_2\,, \tag{D13}$$

where

$$I_1 \equiv \sum_P \frac{1}{P^2}\,, I_2 \equiv \sum_P \frac{1}{(P^2)^2}\,. \tag{D14}$$

After the complete bosonic thermal-IBP reduction, the entire gauge-gauge slope is

$$\left[\Phi_{g(g)}\right]_{k^2} = k^2 C_{\text{gg}}(d) I_1 I_2\,, \tag{D15}$$

with

$$C_{\text{gg}}(d) = \frac{(d-8)(d-3)(d-2)(d-1)^3(d+1)}{12 d (d-5)}\,. \tag{D16}$$

Near the physical dimension,

$$C_{\text{gg}}(3-2\epsilon) = -\frac{40}{9}\epsilon + \frac{652}{27}\epsilon^2 + O(\epsilon^3)\,, \tag{D17}$$

whereas

$$I_2 = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + O(1)\right]. \tag{D18}$$

The finite result is therefore an evanescent-times-divergent contribution. Taking $d=3$ before the Laurent expansion would incorrectly erase it. The exact slope is

$$H_{\mathrm{gg}}=-\frac{5}{108\pi^2}=-0.004690795539. \tag{D19}$$

There is no surviving pole or explicit logarithm in this sector. With $C_{\mathrm{gg}}=0.899790408$,

$$\frac{\delta A_{\mathrm{gg},hard}}{\left(k_B T\right)^2}=-0.001055183207970. \tag{D20}$$

The Yukawa calculation contains mixed bosonic and fermionic Matsubara statistics. One master cancels algebraically before any integration, and the remaining static expression reduces to fourteen $y=1$ zero-momentum structures plus twenty-four genuine $y=0$ Taylor structures. Spatial integration-by-parts identities reduce the complete mixed-statistics problem to nineteen products of one-loop Bose and Fermi tadpoles. Defining

$$B_\alpha^{(r)}\equiv\sum_P\frac{P_0^r}{\left(P^2\right)^\alpha},F_\alpha^{(r)}\equiv\sum_{\{P\}}\frac{P_0^r}{\left(P^2\right)^\alpha}, \tag{D21}$$

the exact one-loop relation

$$F_\alpha^{(r)}=\left(2^{2\alpha-r-d}-1\right)B_\alpha^{(r)} \tag{D22}$$

converts the mixed sector into the same analytic thermal-tadpole basis. Its Laurent expansion is

$$H_y=\frac{1}{72\pi^2\epsilon}+\frac{4\ln\left(\acute{\mu}/2\pi k_B T\right)-24\zeta'\left(-1\right)-6\ln 2-1+2\gamma_E}{72\pi^2}+O\left(\epsilon\right). \tag{D23}$$

At the central scale,

$$H_y^{\overline{MS}}=-4.83306334474441\times10^{-5}, \tag{D24}$$

and with $C_y=0.121181896$,

$$\frac{\delta A_{y,hard}}{\left(k_B T\right)^2}=-0.000001464199449. \tag{D25}$$

The scalar-gauge sector supplies an independent check of the bosonic reduction. Twenty-nine combined masters reduce by static power counting to fifteen $y=1$ and seven $y=0$ candidates. Three of those candidates vanish identically after the exact two-loop factorization identities are applied. The entire surviving sector again collapses to the same tadpole product as the Higgs self-coupling piece,

$$H_{sg}=C_{sg}\left(d\right)I_1 I_2, \tag{D26}$$

with

$$C_{sg}(d)=\frac{(d-2)(d-1)(d+1)(2d^3-17d^2+35d-72)}{12d(d-5)}. \tag{D27}$$

Equivalently,

$$H_{sg}=R_{sg}(d)H_{\lambda},R_{sg}(d)=\frac{2d^3-17d^2+35d-72}{24d(d-5)}. \tag{D28}$$

Expanding around $d=3-2\epsilon$ gives

$$H_{sg}=\frac{11}{288\pi^2\epsilon}+\frac{-528\zeta'(-1)+88\ln(\acute{\mu}/4\pi k_B T)-49+44\gamma_E}{576\pi^2}+O(\epsilon). \tag{D29}$$

At $\acute{\mu}=2\pi k_B T$,

$$H_{sg}^{\overline{MS}}=0.000482489583809332, \tag{D30}$$

and with $C_{sg}=0.246368054$,

$$\frac{\delta A_{sg,hard}}{(k_B T)^2}=+0.000029717504960. \tag{D31}$$

The fermion-gauge sector is the largest algebraic reduction. The three published master families contain different Bose/Fermi frequency assignments; after identical structures are combined there are ninety-seven unique masters. Static power counting removes all ten $y=2$ terms and fourteen $y=0$ terms with no surviving external propagator, leaving

$$33\ y=1+40\ y=0=73 \tag{D32}$$

candidate structures. By mapping the three Matsubara assignments to a common pair of fermionic loop variables, the entire problem becomes one canonical mixed-statistics two-loop vacuum calculation. The static algebra initially contains 141 canonical terms, but thermal IBP reduces them to only nineteen one-loop tadpole products.

This sector provides the sharpest ultraviolet consistency check. Individual factorized pieces contain both simple and double poles, yet the complete combination satisfies

$$H_{fg}|_{1/\epsilon^2}=0,H_{fg}|_{1/\epsilon}=0. \tag{D33}$$

The finite remainder reduces first to trigamma functions,

$$H_{fg}=\frac{-60\psi^{(1)}(-1/2)-16+6\psi^{(1)}(3/2)+27\pi^2}{17280\pi^2}, \tag{D34}$$

and the exact identities

$$\psi^{(1)}\left(-\frac{1}{2}\right)=\frac{\pi^2}{2}+4,\psi^{(1)}\left(\frac{3}{2}\right)=\frac{\pi^2}{2}-4 \tag{D35}$$

give

$$H_{fg} = -\frac{7}{432\pi^2} = -0.001641778438648992. \tag{D36}$$

No explicit $\acute{\mu}$ dependence remains. With $C_{fg} = 2.274022931$,

$$\frac{\delta A_{fg,hard}}{(k_B T)^2} = -0.000933360454277. \tag{D37}$$

The five finite strict-hard operator contributions at the central scale are summarized in Table D1.

Table D1. Finite strict-hard Standard-Model contributions to the static stiffness at the central scale.

hard sector	$\delta A_{hard}/(k_B T)^2$
Higgs self-coupling	-0.00001510021333
gauge-gauge	-0.00105518320797
Yukawa	-0.00000146419944
scalar-gauge	$+0.00002971750490$
fermion-gauge	-0.00093336045427
strict-hard operator subtotal	-0.00197539057007

Thus

$$\frac{\delta A^{(2)}_{hard,op}}{(k_B T)^2} = -0.001975390570074915. \tag{D38}$$

The size hierarchy is useful physically. The gauge-gauge and fermion-gauge pieces dominate the finite hard correction; the scalar-gauge term is positive but much smaller, while the Higgs self-coupling and Yukawa pieces are numerically tiny at the controlling high-state couplings. The net strict-hard operator term lowers the $\xi_H = 0$, $O(1) + O(g)$ benchmark by less than one percent. Its importance lies as much in its renormalization structure as in its numerical size.

The operator subtotal is not yet the complete metric-variational answer. The Higgs self-coupling, Yukawa and scalar-gauge sectors retain a combined ultraviolet and explicit-scale ledger. In the OQG normalization the strict-hard operator contribution contains

$$\left.\frac{A^{(2)}_{hard,op}}{(k_B T)^2}\right|_{1/\epsilon} = \frac{0.000263263990494829}{\epsilon}, \tag{D39}$$

and

$$\frac{\partial}{\partial \ln \acute{\mu}} \left(\frac{A^{(2)}_{hard,op}}{(k_B T)^2} \right) = 0.001053055961979316. \tag{D40}$$

The gauge-gauge and fermion-gauge sectors contribute neither pole nor explicit logarithm to this residual ledger. The remaining terms are not removed by hand. They are precisely the structures that the hard Higgs curvature contact and the dimensionally reduced electric/Higgs and local-curvature matching must complete. Appendix E carries out that variational completion and shows that the combined hard-plus-soft result reproduces the required ξ_H running and closes the two-scale renormalization problem.

The strict-hard calculation therefore establishes a complete analytic foundation for the perturbative part of Sec. 5. Every one of the five Standard-Model interaction classes has been reduced to explicit thermal sum-integrals and evaluated; the largest sectors contain nontrivial cancellations that survive independent numerical checks; and no unidentified hard master integral remains. The hard result is deliberately handed to the soft matching with its residual pole and scale dependence still visible, so the final renormalization closure is a prediction of the completed Standard-Model Hessian rather than an imposed subtraction.

Appendix E. Three-dimensional electric/Higgs and local-R matching

Appendix E completes the perturbative matching that Appendix D intentionally leaves unfinished. At $O\left(g^2\right)$, the same dimension-two susceptibility receives contributions from several momentum ranges, and no one range is meaningful by itself. Dimensional reduction handles this by integrating out the hard nonzero Matsubara modes and describing the slower bosonic zero modes with a three-dimensional effective theory.

The reader can follow the appendix as one cancellation problem. The hard Higgs contact, the screened electric/Higgs sector, the local curvature Wilson coefficient, and the Ward-required contacts must combine so that ultraviolet poles and arbitrary matching-scale logarithms disappear from the completed Hessian. The result is that all perturbative ambiguity closes without an independent gravitational parameter. What remains at the end is genuinely different physics: the finite confining magnetostatic Yang-Mills contribution treated in Appendix F.

To keep the dimensional-reduction formulas compact, this appendix writes T for the thermal energy $k_B T_{phys}$. Restoring International System of Units (SI) thermodynamic notation amounts simply to replacing every occurrence of T in an energy ratio by $k_B T$. The four-dimensional modified-minimal-subtraction scale is $\acute{\mu}$, while μ_3 denotes the independent three-dimensional modified-minimal-subtraction scale. The couplings g_Y, g_2, g_3 are the hypercharge, weak-isospin, and strong gauge couplings, λ is the Higgs quartic coupling, and $Y_2 \equiv 3\left(y_t^2 + y_b^2\right) + y_\tau^2$ is the retained Yukawa invariant. NLO means next-to-leading order, EFT means effective field theory, and RG means renormalization group. The hard/electric/magnetostatic separation follows high-temperature Standard-Model dimensional reduction as developed by Kajantie, Laine, Rummukainen and Shaposhnikov (and the equivalent EFT organization of Braaten and Nieto) [4,14].

Appendix D leaves the complete strict-hard operator subtotal

$$\frac{\delta A^{(2)}_{hard,op}}{T^2} = -0.001975390570074915, \tag{E1}$$

but the static OQG observable is the full metric-variational Hessian. The curved Standard Model contains the nonminimal Higgs operator $\xi_H R H^\dagger H$, whose second metric variation contributes the local TT contact

$$A_{H,contact} = \frac{\xi_H}{2} \langle H^\dagger H \rangle. \tag{E2}$$

The hard interaction correction to the Higgs condensate is taken from the hard/soft/ultrasoft Standard-Model condensate matching of Laine and Meyer [18]:

$$\frac{\delta^{(2)}_{hard} \langle H^\dagger H \rangle}{T^2} = -\frac{0.0015601439258488485}{\epsilon} - 0.005671963506883654, \tag{E3}$$

at the central four-dimensional scale $\acute{\mu} = 2\pi T$. Consequently

$$\frac{\delta A^{(2)}_{H,contact}}{T^2} = -\frac{0.0007800719629244424\,\xi_H}{\epsilon} - 0.002835981753441827\,\xi_H. \tag{E4}$$

The known four-dimensional variational subtotal is therefore

$$\frac{\delta A^{(2)}_{hard,var,known}}{T^2} = -0.001975390570074915 - 0.002835981753441827\,\xi_H. \tag{E5}$$

This quantity is deliberately not finite by itself. The hard/soft split introduced by dimensional reduction is a factorization device, so poles and scale logarithms are allowed to reside in each momentum region provided that the completed Standard-Model response is independent of the arbitrary split.

The curved-space running of the Higgs improvement supplies a stringent check; the required Standard-Model curvature beta function is the one derived by Markkanen, Nurmi, Rajantie and Stopyra [19]. At the working order,

$$16\pi^2 \beta_{\xi_H} = \left(\xi_H - \frac{1}{6}\right)\left[12\lambda + 2Y_2 - \frac{3}{2} g_Y^2 - \frac{9}{2} g_2^2\right]. \tag{E6}$$

For the central running state this requires

$$D_{required} \equiv \frac{\partial}{\partial \ln \acute{\mu}} \left[\frac{A^{(2)}_{2,E}}{T^2}\right]_{explicit} = -0.000080194657458 + 0.000481167944750\,\xi_H. \tag{E7}$$

The hard terms alone instead give

$$D_{hard,known} = 0.001053055961979 - 0.002639119906948\,\xi_H, \tag{E8}$$

so the dimensionally reduced theory is handed the exact matching target

$$D_{remainder} = -0.001133250619438 + 0.003120287851698\,\xi_H . \tag{E9}$$

This target is not an adjustable condition. It is fixed before any three-dimensional finite constant is evaluated.

The electrostatic Standard Model contains the Higgs zero mode and the temporal gauge scalars. The relevant massive fields are

$$H\,,\, A_0^a\left(SU\left(2\right)\right), B_0\,, \tag{E10}$$

with leading potential interactions

$$V_{3D} \supset \lambda_3\left(H^\dagger H\right)^2 + h_3 H^\dagger H\, A_0^a A_0^a + h_3{}'\, H^\dagger H\, B_0^2\,, \tag{E11}$$

where

$$\lambda_3 = \lambda T\,, h_3 = \frac{1}{4} g_2^2 T\,, h_3{}' = \frac{1}{4} g_Y^2 T\,. \tag{E12}$$

For one real massive three-dimensional scalar with curvature coupling ξ, the coincident propagator has the local expansion

$$G\left(x\,,x\right) = -\frac{m}{4\pi} + \frac{\left(1/6-\xi\right)R}{8\pi m} + O\left(R^2\right), \tag{E13}$$

so its one-loop curvature coefficient is

$$a_R^{\left(1\right)} = \frac{m}{8\pi}\left(\frac{1}{6} - \xi\right). \tag{E14}$$

With the OQG normalization

$$A_{2,E}^{\left(3D\right)} = \frac{T}{2} a_R\,, \tag{E15}$$

Eq. (E14) reproduces the zero-mode Kubo coefficient derived in Appendix C. This is the normalization anchor for the two-loop soft theory.

The analytically separable scalar-potential diagrams are finite. For the Higgs self-quartic,

$$\frac{\delta A_{\lambda_3}}{T^2} = -\frac{\lambda}{32\pi^2}\left(1 - 6\,\xi_H\right), \tag{E16}$$

which at the central state is

$$\frac{\delta A_{\lambda_3}}{T^2} = +0.000106345651105 - 0.000638073906633\,\xi_H\,. \tag{E17}$$

For N_X real electric scalars X_A coupled through $h_X H^\dagger H\, X_A X_A$, the one-vertex figure-eight gives

$$\frac{\delta A_{HX}}{T^2} = -\frac{(h_X/T)N_X}{32\pi^2}\left[\left(\frac{1}{6}-\xi_H\right)\frac{m_X}{m_H}+\frac{1}{6}\frac{m_H}{m_X}\right]. \tag{E18}$$

The hypercharge-electric and weak-electric terms are respectively

$$\frac{\delta A_{HB_0}}{T^2} = -0.000080238124308 + 0.000399930611425\,\xi_H, \tag{E19}$$

and

$$\frac{\delta A_{HA_0}}{T^2} = -0.000275760532043 + 0.001397326778331\,\xi_H. \tag{E20}$$

Adding the three potential contributions gives

$$\frac{\delta A^{(2)}_{3D,pot}}{T^2} = -0.000249653005245 + 0.001159183483123\,\xi_H. \tag{E21}$$

The same result follows independently by computing the soft one-loop mass shifts and inserting them into Eq. (E14). That agreement is more than a numerical convenience: it makes explicit why inserting a separately computed NLO screening mass into the old ring term and then also evaluating the EFT diagrams would double count the same soft physics. The entire potential block is finite at this order,

$$D^{(2)}_{3D,pot} = 0, \tag{E22}$$

so the renormalization ledger of Eq. (E9) must be carried by the derivative/gauge, Higgs-contact and local-curvature pieces.

The derivative/gauge block involves every charged scalar zero mode. In addition to the weak-electric adjoint and the Higgs, the color-electric adjoint $A_0^{SU(3)}$ contributes even though the Higgs is color neutral. There are no fermionic Matsubara zero modes. The ultraviolet coefficients of the electric operator sector are labeled as follows: D_{E3} is the color-electric $SU(3)$ coefficient, D_{E2} the weak-electric $SU(2)$ coefficient, and D_{EY} the hypercharge-electric coefficient. Their magnetostatic counterparts D_{M3} and D_{M2} are the $SU(3)$ and $SU(2)$ spatial-gauge logarithmic coefficients. The three-dimensional spin-two ultraviolet/local-curvature structure is anchored by Corianò, Delle Rose and Skenderis, with the pure-Yang-Mills limit independently checked by Caron-Huot, Pokraka and Zahraee [10,11]. The ultraviolet coefficients of the electric operator sector are

$$D^{op}_{E3} = -0.002999279573124\,,\ D^{op}_{E2} = -0.001199069963351\,,\ D^{op}_{EY} = -0.0001203579874 \tag{E23}$$

and therefore

$$D_{E,op} = -0.004318707523974 = -\frac{24g_3^2+9g_2^2+g_Y^2}{192\pi^2}. \tag{E24}$$

The corresponding direct loop pole coefficient in $A_{2,E}/T^2$ is

$$P_{E,op}=-0.001079676880993. \quad \text{(E25)}$$

An independent pure-Yang-Mills calculation supplies the perturbative magnetostatic ultraviolet coefficients

$$D_{M3}=+0.002999279573124, D_{M2}=+0.000799379975567. \quad \text{(E26)}$$

Their role here is only to complete the matching ledger; their finite nonperturbative contribution is deferred to Appendix F. The exact equality

$$D_{E3}^{op}+D_{M3}=0 \quad \text{(E27)}$$

is a useful structural check: the color-electric and color-magnetostatic ultraviolet logarithms cancel in the combined three-dimensional operator response. The remaining soft operator derivative is

$$D_{soft,op}=D_E+D_M=-0.000520047975283=-\frac{g_Y^2+3g_2^2}{192\pi^2}. \quad \text{(E28)}$$

The Higgs contact closes the part of the ledger carrying ξ_H. The matched three-dimensional condensate contains the opposite pole to Eq. (E4),

$$\left.\frac{\delta A_{H,contact}^{(2)}}{T^2}\right|_{3D,pole}=+\frac{0.000780071962924\,\xi_H}{\epsilon}, \quad \text{(E29)}$$

so the hard and 3D contact poles cancel exactly. Its explicit logarithmic derivative is

$$D_{H,contact}^{3D}=+0.003120287851698\,\xi_H. \quad \text{(E30)}$$

Together with the hard contact derivative,

$$-0.002639119906948\,\xi_H+0.003120287851698\,\xi_H=0.000481167944750\,\xi_H, \quad \text{(E31)}$$

which is precisely the ξ_H-dependent coefficient required by Eq. (E7). Thus the complete improvement-coupling RG ledger closes before any value of ξ_H is chosen.

The finite contact is equally well behaved. Let

$$c_\xi=\frac{g_Y^2+3g_2^2}{2(4\pi)^2}=0.003120287851698. \quad \text{(E32)}$$

The full-theory matching logarithm and the renormalized 3D condensate are complementary pieces,

$$\frac{\delta A_{H,match}}{T^2}=c_\xi\,\xi_H\ln\frac{\acute{\mu}}{\mu_3}, \quad \text{(E33)}$$

$$\frac{\delta A_{H,3D,finite}}{T^2}=c_\xi \xi_H\left[\ln\frac{\mu_3}{2m_H}+\frac{1}{4}\right]. \tag{E34}$$

Their sum is independent of the intermediate 3D scale,

$$\frac{\delta A^{(2)}_{H,gauge,matched}}{T^2}=c_\xi \xi_H\left[\ln\frac{\acute{\mu}}{2m_H}+\frac{1}{4}\right]. \tag{E35}$$

At the controlling scales this is

$$\frac{\delta A^{(2)}_{H,gauge,matched}}{T^2}=+0.008194063689158\,\xi_H. \tag{E36}$$

The exact cancellation of μ_3 in Eq. (E35) is an independent finite-level check on the pole and logarithm matching.

The masses retained in the derivative/gauge sector are

$$\frac{m_{E3}}{T}=0.688205816598,\frac{m_{E2}}{T}=0.680335,\frac{m_H}{T}=0.291904. \tag{E37}$$

The matched three-dimensional ultraviolet result in Eqs. (E23)-(E28) fixes the coefficient of the low-momentum mass logarithms. Before the finite threshold constant is evaluated, the massive operator therefore has the form

$$\begin{aligned}\frac{\delta A^{(2)}_{E,op}}{T^2}=&-\frac{1}{192\pi^2}[24g_3^2\ln\frac{\mu_3}{m_{E3}}+6g_2^2\ln\frac{\mu_3}{m_{E2}}\\&+\left(3g_2^2+g_Y^2\right)\ln\frac{\mu_3}{m_H}]+C_{E,loc}.\end{aligned} \tag{E38}$$

At $\mu_3/T=g_2^2$=0.252466052, the fixed mass-log subtotal is

$$\frac{\delta A_{E,\log}}{T^2}=0.003875627218801. \tag{E39}$$

Retaining the unequal Debye and Higgs masses matters. Relative to replacing every electric scalar mass by m_H, the color and weak adjoint thresholds contribute

$$\Delta_3=0.002572371008969,\Delta_2=0.000676403634177, \tag{E40}$$

so the finite mass splitting is

$$\Delta_{mass\,split}=0.003248774643146. \tag{E41}$$

This is a real consequence of the physical screening hierarchy, not a common-scale convention.

The absolute massive low-q constant is obtained most cleanly from the curved three-dimensional effective action. The multiloop Riemann-normal-coordinate heat-kernel reduction follows Carneiro and von Gersdorff, including the nonzero-spin extension needed for the spatial gauge operator [8,9]. For one massive charged scalar coupled to a massless spatial gauge field, the Ward-complete $O\left(g_3^2\right)$ vacuum graphs are the two-current exchange graph and the $A^2\phi^\dagger\phi$ seagull. Expanding the scalar and Feynman-gauge vector heat kernels in Riemann normal coordinates and projecting on the coefficient of R reduces the two-loop problem to

$$J_R(d,m)=(4\pi)^{-d}\Gamma(3-d)\left(m^2\right)^{d-3}I(d), \tag{E42}$$

where

$$I(d)=-\frac{1}{24}\int_0^1 dx\int_0^\infty dy\,\frac{N_d(x,y)}{\left[x(1-x)+y\right]^{d/2+2}}, \tag{E43}$$

with $a=x(1-x)$ and

$$N_d(x,y)=a(d+4a)+d(4a+1)y+(5d-32a-2)y^2+4(d-6)y^3. \tag{E44}$$

The integral reduces completely to beta functions. The two quantities needed for the Laurent expansion are

$$I(3)=-\pi,\ \frac{I'(3)}{I(3)}=2\ln 2-\frac{5}{9}. \tag{E45}$$

With the standard three-dimensional $\overline{MS}$ measure, the result expands as

$$J_R\propto\frac{I(3)}{2\epsilon}+2I(3)\ln\frac{\mu_3}{m}-I'(3)+O(\epsilon). \tag{E46}$$

The direct pole P and explicit scale derivative D therefore obey the independent normalization check

$$D=4P, \tag{E47}$$

and the finite massive scalar-gauge constant is

$$c_{sg}^{(m)}=-\frac{I'(3)}{2I(3)}=\frac{5}{18}-\ln 2=-0.415369402782167. \tag{E48}$$

Thus each massive charged-scalar block contributes

$$\frac{\delta A_{E,i}^{(2)}}{T^2}=D_i\left[\ln\frac{\mu_3}{m_i}+\frac{5}{18}-\ln 2\right]. \tag{E49}$$

A temporary magnetic regulator makes the Ward completion visible. Separately, exchange and seagull contain

$$\Gamma_{exch}^{(R)}\supset+g_3^2G_R\frac{m}{64\pi^2m_M}R, \tag{E50}$$

$$\Gamma_{seagull}^{(R)}\supset-g_3^2G_R\frac{m}{64\pi^2m_M}R, \tag{E51}$$

so

$$\left(\Gamma^{(R)}_{exch}+\Gamma^{(R)}_{seagull}\right)\Big|_{1/m_M}=0. \tag{E52}$$

The electric sector therefore does not hide an additional powerlike magnetostatic contribution. The pure magnetic problem remains cleanly separable.

For the Standard Model, Eq. (E49) gives the complete massive low-q loop contributions summarized in Table E1.

Table E1. Massive three-dimensional electric-sector contributions to the static stiffness at the central matching point.

Electric block	Mass log	Finite threshold term	Complete loop
$A_0^{SU(3)}$	$+0.003007711007125$	$+0.001245808965065$	$+0.004253519972190$
$A_0^{SU(2)}$	$+0.000792432189903$	$+0.000332037983047$	$+0.001124470172950$
Higgs gauge operator	$+0.000075484021773$	$+0.000216012016911$	$+0.000291496038684$

The finite electric constant is therefore

$$C^{\overline{MS}}_{E,loc}=0.001793858965024, \tag{E53}$$

and the complete perturbative massive electric derivative/gauge loop is

$$\frac{\delta A^{(2)}_{E,op}}{T^2}=0.005669486183825. \tag{E54}$$

The massless overlap calculation provides a useful check rather than a replacement for Eq. (E54). Its finite/log ratios are

$$R_E=\frac{8+6\pi^2}{32}=2.100550825204254,\, R_M=\frac{40-6\pi^2}{32}=-0.600550825204255. \tag{E55}$$

Thus the corresponding electric and magnetic overlap constants are

$$C^{ov,\overline{MS}}_E=-0.009071664653299,\, C^{ov,\overline{MS}}_M=-0.002281288126636. \tag{E56}$$

The low-q and overlap electric expressions have the same logarithmic coefficient but different finite constants, as required on opposite sides of the electric threshold. Their difference is

$$D_E\left[c^{(m)}_{sg}-R_E\right]=+0.010865523618323. \tag{E57}$$

This confirms why the massless overlap form factor could not be used to guess the Kubo-limit finite constant.

The last perturbative issue is the local curvature coefficient. Two independent scales must be kept separate: the four-dimensional hard matching scale $\acute{\mu}$ and the intrinsic three-dimensional renormalization scale μ_3. Since

$$D_{soft}=D_E+D_M=-0.000520047975283, \tag{E58}$$

the three-dimensional local coefficient must obey

$$\frac{\partial C_R^{3D}}{\partial \ln \mu_3}=+0.000520047975283. \tag{E59}$$

Independently, matching to the four-dimensional hard theory requires

$$\frac{\partial C_R^{3D}}{\partial \ln \acute{\mu}}=-0.001133250619438. \tag{E60}$$

It is therefore useful to write

$$C_R^{3D}\left(\acute{\mu},\mu_3\right)=D_{soft}\ln\frac{\acute{\mu}}{\mu_3}+D_{R,diag}\ln\frac{\acute{\mu}}{2\pi T}+C_{R,0}^{fin}, \tag{E61}$$

where

$$D_{R,diag}=-0.000613202644155. \tag{E62}$$

The number in Eq. (E62) is the derivative obtained when $\acute{\mu}$ and μ_3 are varied together. It is not a third independent beta function; it is simply

$$D_{R,diag}=-0.001133250619438+0.000520047975283. \tag{E63}$$

The complete second-metric-variation audit fixes the finite boundary. The contact classification follows the metric-variational analysis of Kovtun and Shukla, while the absence of an independent strict-high-state four-dimensional Einstein-term running follows the curved-Standard-Model result of Markkanen, Nurmi, Rajantie and Stopyra [17,19]. In a homogeneous state, the scalar sector's only two-derivative contact is the already included $\xi_H R H^{\dagger} H$ term. The Dirac stress contains no analogous two-metric-derivative contact, the Yang-Mills metric dependence is algebraic because $F_{\mu\nu}$ contains no Levi-Civita connection, and the quartic and Yukawa interactions can produce only momentum-independent seagulls at this order. A pure four-dimensional Einstein term could shift the k^2 Hessian, but its ultraviolet running is proportional to the scalar mass parameter and vanishes in the strict high-state $v^2\to 0$ matching. Any remaining finite state-independent Einstein coefficient cancels from the state-subtracted thermal response rather than becoming a new matter-state parameter. Hence

$$C_{R,0}^{fin}=0 \tag{E64}$$

in the state-subtracted $\overline{MS}$ convention used here.

At the central scales

$$\acute{\mu}=2\pi T,\ \mu_3=g_2^2 T=0.252466052\,T, \tag{E65}$$

the diagonal logarithm vanishes and

$$\ln \frac{\acute{\mu}}{\mu_3} = 3.214355553239699\,. \tag{E66}$$

Therefore

$$C_R^{3D} = -0.001671619097302\,. \tag{E67}$$

The resulting scale cancellations can be checked separately rather than only in the total. At fixed $\acute{\mu}$,

$$\partial_{\ln \mu_3}\left(A_E + A_M + C_R^{3D}\right) = 0\,. \tag{E68}$$

At fixed μ_3, the constant hard derivative plus Eq. (E60) gives exactly the constant part of Eq. (E7), while Eq. (E31) supplies its ξ_H part. Moving the two scales together reproduces Eq. (E62). Thus the hard/soft factorization, intrinsic three-dimensional renormalization and Higgs-improvement running all close simultaneously.

Collecting the hard result of Appendix D, the Higgs variational contact, the finite potential block, the complete massive electric loop and the local curvature bridge gives

$$\frac{A_{known\,through\,electric\,/\,local}}{T^2} = 0.263713812532203 + 0.066623718018839\,\xi_H + \frac{A_M^{NP}\left(\mu_3\right)}{T^2}\,. \tag{E69}$$

Equation (E69) is the endpoint of the perturbative electric/Higgs/local matching calculation. No arbitrary finite gravitational coefficient remains. The Higgs-curvature coupling is retained as the renormalized Standard-Model input that it is, and its complete $O\left(g^2\right)$ RG dependence has closed. The only qualitatively distinct contribution still missing is the finite magnetostatic TT susceptibility of pure three-dimensional $SU\left(3\right)$ and $SU\left(2\right)$ Yang-Mills. Appendix F isolates that problem, shows why existing glueball spectral data do not determine the required local curvature coefficient, and constructs the lattice matching observable that can supply the last two numbers.

Appendix F. Magnetostatic pure-Yang-Mills reduction and lattice matching

Appendix F isolates the only genuinely nonperturbative part of the static calculation. After the hard, electric/Higgs, and local-curvature pieces have been matched, the remaining response comes from the confining *spatial* zero modes of $SU\left(3\right)$ and $SU\left(2\right)$. This is the magnetostatic sector of the high-temperature Standard Model; it is not ordinary electromagnetic magnetism.

Each gauge group reduces to a pure three-dimensional Yang-Mills theory with one intrinsic scale, so the entire unknown finite contribution is encoded in only two dimensionless numbers. The main technical accomplishment is to define those numbers in a way that can be extracted from standard flat-lattice ensembles. An infrared-minus-ultraviolet secant subtraction cancels the additive lattice curvature counterterm, so neither a curved-background simulation nor a separate lattice Einstein-counterterm calculation is required.

As in Appendix E, T denotes the thermal energy $k_B T_{phys}$. The notation $\overline{MS}$ denotes modified minimal subtraction, μ_3 is the three-dimensional $\overline{MS}$ renormalization scale, and the superscript NP means nonperturbative. For $SU(N_c)$, $C_A = N_c$ is the adjoint quadratic Casimir and $d_G = N_c^2 - 1$ is the adjoint representation dimension. The coefficient $D_{M,G}$ denotes the known magnetostatic logarithmic coefficient for gauge group G. After dimensional reduction the two non-Abelian magnetostatic theories have couplings

$$g_{M,3}^2 = g_3^2 T \, , g_{M,2}^2 = g_2^2 T \, , \tag{F1}$$

and for a generic gauge group G it is convenient to define the intrinsic Yang-Mills scale

$$\lambda_G \equiv C_A g_{M,G}^2 . \tag{F2}$$

At the central Standard-Model running state,

$$\frac{\lambda_3}{T} = 3 g_3^2 = 0.710440869000 \, , \frac{\lambda_2}{T} = 2 g_2^2 = 0.504932104000 . \tag{F3}$$

Because the coefficient of a three-dimensional curvature term has mass dimension one and pure Yang-Mills supplies no intrinsic dimensionful quantity other than λ_G, the finite magnetostatic response must be proportional to this scale. The ultraviolet logarithm is already fixed by the two-loop pure-Yang-Mills stress correlator, so the most general renormalized contribution needed here is

$$\frac{A_{M,G}^{NP}(\mu_3)}{T^2} = D_{M,G}\left[\ln\frac{\mu_3}{\lambda_G} + c_G^{NP}\right]. \tag{F4}$$

The Standard-Model group weights inherited from Appendix E are

$$D_{M,SU(3)} = 0.002999279573124 \, , D_{M,SU(2)} = 0.000799379975567 . \tag{F5}$$

Thus the entire nonperturbative magnetostatic content is contained in two pure numbers,

$$c_3^{NP} , c_2^{NP} . \tag{F6}$$

At the central three-dimensional scale $\mu_3 / T = g_2^2 = 0.252466052$,

$$\ln\frac{\mu_3}{\lambda_3} = - 1.034608927432226 \, , \ln\frac{\mu_3}{\lambda_2} = - \ln 2 \, , \tag{F7}$$

so

$$\frac{A_M^{NP}}{T^2} = - 0.003657169398480 + 0.002999279573124\, c_3^{NP} + 0.000799379975567\, c_2^{NP} . \tag{F8}$$

Combining Eq. (F8) with the completed hard/electric/local result of Appendix E gives the full static stiffness in its most compact parameterized form,

$$\frac{A_{2,E}}{T^2} = 0.260056643133723 + 0.066623718018839\,\xi_H + 0.002999279573124\,c_3^{NP} + 0.000799379975567\,c_2^{NP}. \tag{F9}$$

No unknown perturbative coefficient remains in Eq. (F9). The two undetermined quantities are finite observables of pure three-dimensional Yang-Mills in a specified state-subtracted $\overline{MS}$ convention.

The known group factors also show where the numerical leverage resides. Of the total magnetostatic logarithmic weight,

$$\frac{D_{M,SU(3)}}{D_{M,SU(3)} + D_{M,SU(2)}} = 0.789562616676, \tag{F10}$$

so approximately 79% is carried by the color magnetostatic sector and approximately 21% by the weak magnetostatic sector. This is a statement about the known group-theory coefficients only; it does not assume that c_3^{NP} and c_2^{NP} are equal or even similar.

The existing pure-Yang-Mills literature fixes the ultraviolet part of the problem much more strongly than its infrared finite part. Caron-Huot, Pokraka and Zahraee decompose the three-dimensional stress correlator into spin-zero and spin-two form factors [10]. In the spin-two channel their two-loop result contains

$$A_2^{(1)}(p^2) = \lambda_G\, p^2 \left[-1 + \frac{20}{3\pi^2} + \frac{4}{3\pi^2 \epsilon} - \frac{8}{3\pi^2} \ln \frac{p^2}{\mu_3^2} \right], \tag{F11}$$

up to the common tensor normalization. This fixes the p^2/ϵ divergence, the logarithm and the gauge-group factor. In the OQG normalization it reproduces exactly Eq. (F5). Only the finite infrared boundary generated by confinement remains undetermined.

The same published work also explains why glueball masses and stress-tensor residues do not supply that missing boundary. In three dimensions the superconvergent combination is

$$A_{0+2}(p^2) = \frac{A_0(p^2) + A_2(p^2)}{(p^2)^2}. \tag{F12}$$

The local p^2 terms cancel in this combination. That cancellation is precisely what makes the observable scheme independent and permits an unsubtracted positive spectral representation, but it also removes the local curvature information required by the present Kubo coefficient. A narrow glueball contribution therefore enters the reconstructed separated-point tensor form factor as $O(p^4)$ near the origin rather than as the desired p^2 term. Let m_G denote the glueball mass and Z_G its stress-tensor residue in this schematic narrow-state representation. In schematic form,

$$A_2^G(p^2) \propto p^4 \frac{2 Z_G^2 m_G^4}{p^2 + m_G^2} = O(p^4)\,(p \to 0). \tag{F13}$$

The glueball spectrum is consequently indispensable for confirming the gapped confining infrared theory, but it is intentionally blind to the finite local Einstein/curvature coefficient. The required observable is the full metric-variational shear response itself.

For one pure Yang-Mills group, introduce the Euclidean action in a weak spatial metric,

$$S_G[g]=\frac{1}{4g_M^2}\int d^3x\sqrt{g}\,g^{ik}g^{jl}F_{ij}^aF_{kl}^a, \tag{F14}$$

and choose the static off-diagonal shear

$$g_{ij}(z)=\begin{pmatrix}1 & h(z) & 0\\ h(z) & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, |h|\ll 1. \tag{F15}$$

Writing

$$F_{12}^a=a^a, F_{13}^a=b^a, F_{23}^a=c^a,$$

the Yang-Mills density expands through second order as

$$\frac{\sqrt{g}}{4g_M^2}g^{ik}g^{jl}F_{ij}^aF_{kl}^a=\frac{a^2+b^2+c^2}{2g_M^2}-\frac{h}{g_M^2}b^ac^a+\frac{h^2}{4g_M^2}\left(a^2+b^2+c^2\right)+O\left(h^3\right). \tag{F16}$$

The first metric derivative therefore gives the shear stress exactly,

$$T_{12}=\frac{1}{g_M^2}F_{13}^aF_{23}^a, \left.\frac{\delta S}{\delta h(x)}\right|_{h=0}=-T_{12}(x), \tag{F17}$$

and the second derivative gives the Ward-required microscopic contact,

$$\left.\frac{\delta^2 S}{\delta h(x)^2}\right|_{h=0}=L_{YM}(x)=\frac{1}{4g_M^2}F_{ij}^aF_{ij}^a. \tag{F18}$$

This result is important for both the continuum and lattice constructions. The h_{12} source is a mixed orientation $F_{13}F_{23}$ operator; it is not equivalent to assigning different scalar weights to axis-aligned plaquettes. A faithful lattice realization must therefore contain a gauge-invariant cross-orientation source, such as a clover field-strength product or an equivalent metric-aware loop construction.

Define

$$Z[h]=\int DU\,e^{-S[h]}, W[h]=-\ln Z[h]. \tag{F19}$$

For a source amplitude η,

$$\frac{d^2W}{d\eta^2}=\left\langle\frac{d^2S}{d\eta^2}\right\rangle-\left\langle\left(\frac{dS}{d\eta}\right)^2\right\rangle_c. \tag{F20}$$

Using complex Fourier components h_k, let V denote the three-dimensional spatial volume and define the conventional positive-correlator response

$$G_G(k) \equiv -\frac{1}{V} \frac{\partial^2 W}{\partial h_k \partial h_{-k}} |_{h=0}. \tag{F21}$$

With

$$S_k^{(1)} = -\int d^3 x\, e^{ikz} T_{12}(x),\, S_{k,-k}^{(2)} = \int d^3 x\, L_{YM}(x),$$

the completed metric Hessian is

$$G_G(k) = \frac{1}{V} \left[\langle S_k^{(1)} S_{-k}^{(1)} \rangle_c - \langle S_{k,-k}^{(2)} \rangle \right]. \tag{F22}$$

Equivalently,

$$G_G(k) = \int d^3 x\, e^{ikz} \langle T_{12}(x) T_{12}(0) \rangle_c - \langle L_{YM} \rangle. \tag{F23}$$

The contact in Eq. (F23) is momentum independent on the flat background. It is part of the correct variational response and is needed for the Ward identity, but it will cancel automatically from the momentum-step observable introduced below.

The relation between the shear slope and a local curvature coefficient can be fixed geometrically, without perturbation theory. For the metric in Eq. (F15),

$$\int d^3 x \sqrt{g} R = -\frac{1}{2} \int d^3 x \left(\partial_z h\right)^2 + O\left(h^3\right), \tag{F24}$$

up to a periodic total derivative. With

$$h(z) = \eta \cos kz,$$

this becomes

$$\int d^3 x \sqrt{g} R = -\frac{1}{4} V \eta^2 k^2 + O\left(\eta^3\right). \tag{F25}$$

A local term $a_R \int \sqrt{g} R$ therefore contributes exactly

$$G_R(k) = a_R k^2. \tag{F26}$$

The desired renormalized magnetostatic quantity is consequently the k^2 coefficient of the full variational response,

$$G_G^{\overline{MS}}(k; \mu_3) = G_G(0) + C_G^{\overline{MS}}(\mu_3) k^2 + O\left(k^4\right). \tag{F27}$$

The coefficient $C_G^{\overline{MS}}$ has mass dimension one. For momentum along z, the scalar projector does not contribute to the $12,12$ component while the spin-two projector has unit $12,12$ component. The published Yang-Mills normalization then implies

$$\frac{\partial C_G^{\overline{MS}}}{\partial \ln \mu_3} = \frac{d_G \lambda_G}{96\pi^2}. \tag{F28}$$

The symmetric OQG metric-source normalization is one half of the conventional correlator normalization, and dimensional reduction therefore gives the direct conversion

$$\frac{A_{M,G}^{NP}}{T^2} = \frac{C_G^{\overline{MS}}(\mu_3)}{2T}. \tag{F29}$$

Differentiating Eq. (F29) with respect to $\ln \mu_3$ reproduces Eq. (F5), which independently locks the normalization.

Equations (F4) and (F29) imply

$$C_G^{\overline{MS}}(\mu_3) = \frac{d_G \lambda_G}{96\pi^2}\left[\ln \frac{\mu_3}{\lambda_G} + c_G^{NP}\right], \tag{F30}$$

so the two unknown numbers follow directly from the continuum shear slopes,

$$c_G^{NP} = \frac{96\pi^2}{d_G \lambda_G} C_G^{\overline{MS}}(\mu_3) - \ln \frac{\mu_3}{\lambda_G}. \tag{F31}$$

For the Standard-Model groups,

$$c_3^{NP} = 12\pi^2 \frac{C_3^{\overline{MS}}}{\lambda_3} - \ln \frac{\mu_3}{\lambda_3}, \lambda_3 = 3 g_{M,3}^2, \tag{F32}$$

$$c_2^{NP} = 32\pi^2 \frac{C_2^{\overline{MS}}}{\lambda_2} - \ln \frac{\mu_3}{\lambda_2}, \lambda_2 = 2 g_{M,2}^2. \tag{F33}$$

At the central scale these become

$$c_3^{NP} = 12\pi^2 \frac{C_3^{\overline{MS}}}{\lambda_3} + 1.034608927432226, \tag{F34}$$

and

$$c_2^{NP} = 32\pi^2 \frac{C_2^{\overline{MS}}}{\lambda_2} + \ln 2. \tag{F35}$$

Equivalently, the two magnetostatic slopes can be inserted into the Standard-Model stiffness without introducing the intermediate c_G^{NP} notation at all,

$$\frac{A_M^{NP}}{T^2} = \frac{C_3^{\overline{MS}} + C_2^{\overline{MS}}}{2T}, \tag{F36}$$

and hence

$$\frac{A_{2,E}}{T^2}=0.263713812532203+0.066623718018839\,\xi_H+\frac{C_3^{\overline{MS}}+C_2^{\overline{MS}}}{2T}. \tag{F37}$$

The remaining question is therefore not how to define the finite coefficient, but how to measure $C_G^{\overline{MS}}$ nonperturbatively without importing lattice-regulator contact ambiguities.

A finite background shear does not have to be generated. Let a denote the lattice spacing and L the box linear size, so $V=L^3$ for the cubic ensembles considered below. Because the observable is defined by derivatives at $h=0$, ordinary flat-space pure-Yang-Mills ensembles are sufficient. Using a clover or improved field-strength operator,

$$T_{12}^L(x)=\frac{1}{g_M^2}\,tr\left[F_{13}^L(x)\,F_{23}^L(x)\right], \tag{F38}$$

and

$$L_{YM}^L(x)=\frac{1}{4\,g_M^2}\,tr\left[F_{ij}^L(x)\,F_{ij}^L(x)\right], \tag{F39}$$

one measures the Fourier insertions

$$S_{k,L}^{(1)}=-a^3\sum_x e^{ikz}\,T_{12}^L(x)\,,\,S_{k,-k,L}^{(2)}=a^3\sum_x L_{YM}^L(x)\,, \tag{F40}$$

and the lattice variational response

$$G_G^L(k\,;a\,,L)=\frac{1}{V}\left[\langle\, S_{k,L}^{(1)}S_{-k,L}^{(1)}\rangle_c-\langle\, S_{k,-k,L}^{(2)}\rangle\right]. \tag{F41}$$

A gradient-flow energy-momentum tensor provides an independent normalization and continuum cross-check, following Suzuki’s construction and the two-point implementation of Kitazawa, Iritani, Asakawa and Hatsuda [16,23]. If that route is used, let t denote the gradient-flow time; the continuum limit is taken at fixed positive t before the zero-flow-time limit.

An absolute determination of the small-k slope from Eq. (F41) would ordinarily be awkward because the lattice regulator permits an additive local curvature coefficient. Schematically,

$$G_G^L(k)=G_G^L(0)+\left[C_R^L(a)+C_{G,nonlocal}^L(k)\right]k^2+\cdots, \tag{F42}$$

where

$$C_R^L(a)\sim\frac{\alpha_G}{a}+\beta_G\,\lambda_G\ln\left(a\,\lambda_G\right)+\gamma_G\,\lambda_G+\cdots. \tag{F43}$$

The $1/a$ term is allowed because the three-dimensional Einstein coefficient has dimension one. A direct absolute slope would therefore require separate lattice perturbation theory for the power divergence, logarithm and finite lattice-to-$\overline{MS}$ constant. That entire problem can be eliminated algebraically.

Define a finite-momentum secant in k^2,

$$S_G^L(k_a, k_b) \equiv \frac{G_G^L(k_a) - G_G^L(k_b)}{\hat{k}_a^2 - \hat{k}_b^2}, \hat{k} = \frac{2}{a} \sin \frac{ak}{2}. \tag{F44}$$

Every additive local curvature coefficient enters Eq. (F44) as the same momentum-independent constant,

$$S_G^L(k_a, k_b) = C_R^L(a) + S_{G,nonlocal}^L(k_a, k_b) + O(a^2 k^2). \tag{F45}$$

Choose one infrared pair (k_{i1}, k_{i2}) and one ultraviolet pair (k_{u1}, k_{u2}), both on the same lattice, and form

$$\Delta S_G^L = S_G^L(k_{i1}, k_{i2}) - S_G^L(k_{u1}, k_{u2}). \tag{F46}$$

Then

$$C_R^L(a) \text{cancels identically}. \tag{F47}$$

The cancellation in Eq. (F47) simultaneously removes the power-divergent lattice Einstein coefficient, its logarithm, its unknown finite lattice-scheme constant, the momentum-independent microscopic seagull and the momentum-independent vacuum-energy contact. Only the physical change in the nonlocal response between infrared and ultraviolet momenta remains, together with discretization effects that vanish in the continuum limit.

Confinement supplies the infrared anchor. Because the spin-two spectrum is gapped and the separated-point glueball contribution begins at k^4,

$$\lim_{k_{i1}, k_{i2} \to 0} S_G^{\overline{MS}}(k_{i1}, k_{i2}) = C_G^{\overline{MS}}(\mu_3). \tag{F48}$$

The first nonlocal correction is analytic and quadratic in the small infrared momenta. At the ultraviolet end, choose momenta in the window

$$\lambda_G \ll k_u \ll a^{-1}, \tag{F49}$$

where the same secant is controlled by continuum perturbation theory. The absolute renormalized infrared coefficient is then

$$\begin{aligned} C_G^{\overline{MS}}(\mu_3) &= S_{G,pert}^{\overline{MS}}(k_{u1}, k_{u2}; \mu_3) \\ &+ \lim_{\substack{a \to 0 \\ L \to \infty \\ k_{i1}, k_{i2} \to 0}} \left[S_G^L(k_{i1}, k_{i2}) - S_G^L(k_{u1}, k_{u2}) \right]. \end{aligned} \tag{F50}$$

Equation (F50) is the central lattice-to-$\overline{MS}$ result. It converts a difference of lattice slopes into the absolute continuum local-curvature coefficient while never requiring the additive lattice Einstein counterterm itself to be computed.

The ultraviolet anchor in Eq. (F50) is also known analytically beyond the order required to fix the logarithm. Through three loops, the pure-Yang-Mills spin-two response is

$$G^{\overline{MS}}_{G,pert}(k)=\frac{d_G}{512}\{k^3+\lambda_G k^2\left[-1+\frac{20}{3\pi^2}-\frac{8}{3\pi^2}\ln\frac{k^2}{\mu_3^2}\right]$$
$$+\lambda_G^2 k\left[\frac{431}{768}-\frac{37}{9\pi^2}\right]+\cdots\}+\text{constant contact}. \tag{F51}$$

Defining

$$c_1=-1+\frac{20}{3\pi^2}=-0.324525442384415\,,c_2=\frac{431}{768}-\frac{37}{9\pi^2}=0.144655272803723\,, \tag{F52}$$

the required continuum secant is

$$S^{\overline{MS}}_{G,pert}(k_a,k_b)=\frac{d_G}{512}[$$
$$\frac{k_a^2+k_a k_b+k_b^2}{k_a+k_b}+\lambda_G c_1$$
$$-\frac{8\lambda_G}{3\pi^2}\frac{k_a^2\ln\left(k_a^2/\mu_3^2\right)-k_b^2\ln\left(k_b^2/\mu_3^2\right)}{k_a^2-k_b^2} \tag{F53}$$
$$+\frac{\lambda_G^2 c_2}{k_a+k_b}]+\cdots.$$

The constant contact drops out exactly. In practice the ultraviolet matching pair should be varied within the window of Eq. (F49); stability of the extracted $C_G^{\overline{MS}}$ against that variation is the direct test that perturbation theory and lattice cutoff effects are simultaneously under control.

The limiting procedure must preserve that separation of scales. At fixed physical $L\lambda_G$ and k/λ_G, first take

$$a\lambda_G\to 0. \tag{F54}$$

The use of $\hat{k}$ in Eq. (F44) removes the leading tree-level momentum distortion, and the remaining continuum extrapolation should include the expected $O\left(a^2k^2\right)$ corrections for an $O\left(a^2\right)$ -accurate source. If gradient flow is used, maintain

$$a\ll\sqrt{8t}\ll min\left(\lambda_G^{-1},k^{-1}\right)$$

and take $a\to 0$ at fixed $t>0$ before $t\to 0$.

After the continuum limit is controlled, take

$$L\lambda_G\to\infty. \tag{F55}$$

Let m_{gap} denote the lightest confining mass gap. The gapped spectrum implies ordinary finite-volume corrections proportional to $e^{-m_{gap}L}$, while winding-flux states provide an additional finite-volume contamination that must be monitored. The high-precision 2+1-dimensional $SU(N)$

spectroscopy of Athenodorou and Teper provides an established practical benchmark for demonstrating that such effects are negligible [2].

Only then should the infrared pair be extrapolated to zero momentum,

$$k_{i1}, k_{i2} \to 0. \tag{F56}$$

Because the separated-point response starts at k^4, let b_G denote the finite analytic infrared coefficient controlling the leading k^4 response, equivalently the quadratic momentum dependence of the secant. The corresponding secant has the form

$$S_G^{IR}\left(k_{i1}, k_{i2}\right) = C_G^{\overline{MS}} + b_G\left(k_{i1}^2 + k_{i2}^2\right) + O\left(k_i^4\right). \tag{F57}$$

Using nonzero momenta avoids mixing the desired local curvature slope with the special thermodynamic/global-modulus information carried by an exactly uniform shear on a finite torus.

The statistical observable can be simplified one step further. On each flat ensemble configuration, compute

$$\widetilde{T}_{12}(k) = a^3 \sum_x e^{ikz} T_{12}^L(x), \tag{F58}$$

and the connected momentum-space correlator

$$C_G^L(k) = \frac{1}{V}\left\langle \widetilde{T}_{12}(k)\widetilde{T}_{12}(-k)\right\rangle_c. \tag{F59}$$

Because the explicit second-variation contact is independent of k, it cancels in the momentum differences entering Eq. (F44). Thus the formal derivation retains the full Ward contact, while the principal Monte Carlo measurement can be constructed from connected T_{12} correlators alone, provided the lattice shear operator is normalized as the derivative of the same lattice source action. This is why no curved-metric ensemble is required: the background shear is a generating device whose derivatives are measurable as zero-background insertions.

The resulting determination is overconstrained in useful ways. The continuum extrapolation, volume extrapolation, infrared momentum extrapolation and ultraviolet matching window must each form a stable plateau. Changing the discretization of the shear operator must leave the continuum answer unchanged, and variation of the continuum renormalization scale must reproduce the known running

$$\frac{\partial C_G^{\overline{MS}}}{\partial \ln \mu_3} = \frac{d_G \lambda_G}{96\pi^2}. \tag{F60}$$

These checks convert what would otherwise be a delicate local-contact observable into a quantitatively falsifiable lattice matching calculation.

The magnetostatic result is sometimes tempting to connect to an applied laboratory magnetic field, but that would mix distinct physics. The sectors in this appendix are the non-Abelian

$SU(3)$ color and $SU(2)$ weak spatial zero modes of the high-state Standard Model. At this order there is no analogous pure self-interacting $U(1)$ magnetostatic contribution. The constants c_3^{NP} and c_2^{NP} therefore do not constitute an electromagnetic B-field or Weber control law; they are universal finite numbers of the two confining three-dimensional gauge theories.

Appendix F completes the analytic definition of the static two-point problem. The Standard-Model stiffness has been reduced to explicitly matched perturbative terms, the renormalized Higgs-curvature input ξ_H, and two finite pure-Yang-Mills numbers. Their ultraviolet normalization, metric contact, geometric R-normalization, lattice observable, continuum conversion, order of limits and $\overline{MS}$ extraction are all fixed. What is absent is only the numerical evaluation of c_3^{NP} and c_2^{NP} from genuine nonperturbative ensemble data; no further perturbative Standard-Model matching coefficient is hidden behind them.

Appendix G. Finite-(ω, k) retarded TT kernel and causal topology

Appendix G is the proof layer for the causal picture summarized in Secs. 6 and 7. It proceeds in five steps: construct the finite-(ω, k) TT response, identify the two-particle-production and Landau/scattering continua, continue the response onto the physical resonance sheet, verify the pole numerically and under interaction deformations, and finally separate that ordinary resonance from the analytic Lambert-W benchmark and the proposed strong-field boundary.

A few analytic terms are worth fixing before the derivation. A **branch cut** represents a continuum of states; a **sheet** labels the analytic branch reached when the response is continued around that cut; a **pole** is a discrete resonance; and an **exceptional point** would require a non-Hermitian eigenvalue degeneracy with coalescing eigenvectors (in the scalar zero-forming problem, a double-root condition). The calculation below finds a simple underdamped pole, not such an exceptional point.

The free real-time scalar, fermion, and gauge-field structures are anchored by Ota, Zhu and Zhu and by Brandt and Frenkel. The complete general-momentum Standard-Model TT algebra and the general-kinematics thermal spectral masters are those of Ghiglieri, Jackson, Laine and Zhu and Jackson, including Jackson's numerical implementation. Ward-complete interaction logic follows Nachbagauer, Rebhan and Schwarz; kinetic/damping checks use Arnold, Moore and Yaffe and Burgess and Marini; and the quasiparticle-pole interpretation follows Weldon and Moore and Schlusser [1,5,7,12,13,20–22,24].

As in Appendices E and F, write $T \equiv k_B T_{phys}$ for the thermal energy, where T_{phys} is the Kubo-Martin-Schwinger (KMS) temperature and k_B is Boltzmann's constant. Let z denote the complex frequency, let $k \equiv |k|$ denote the real spatial wave-number magnitude, and let c be the locally measured limiting signal speed. Any damping width written as an energy ratio such as Γ / T is understood in energy units, with the factor of $\hbar$ converting the corresponding decay rate to energy. Introduce the dimensionless retarded frequency and spatial wave number

$$u \equiv \frac{\hbar z}{T}, \; y \equiv \frac{\hbar c k}{T}, \tag{G1}$$

where z lies in the complex frequency plane and c is the locally measured limiting signal speed of the settled state. The physical retarded response is analytic for $\operatorname{Im} z > 0$. A KMS state selects a rest frame, so u and y are independent response coordinates even when the microscopic constituents are effectively massless.

The TT channel is the helicity-± 2 projection of the completed Standard-Model stress Hessian constructed in Appendix A and projected explicitly in Appendix B. Let $\Pi^R_{TT}(z, k)$ denote the retarded TT susceptibility, let $\Pi^R_2(z, k)$ denote its common helicity eigenvalue, and let $1_{2\times 2}$ denote the identity on the two-dimensional helicity space. In an isotropic parity-invariant equilibrium state, rotations about the propagation axis prevent mixing of the two opposite helicities, while parity keeps their eigenvalues equal. Thus the physical TT block is proportional to the identity,

$$\Pi_{TT}^{R}(z,k)=\Pi_{2}^{R}(z,k)\,1_{2\times 2}, \tag{G2}$$

and the causal pole problem reduces to the zeros of one complex scalar eigenvalue of the inverse response. The two polarization vectors remain linearly independent; their equality of eigenvalues is a semisimple degeneracy rather than a non-Hermitian exceptional point.

The thermal response has two distinct real-frequency continua. Let ω denote real physical frequency. For positive frequency and massless collisionless constituents, scattering of thermally populated excitations produces the Landau interval

$$0<\omega<ck, \tag{G3}$$

while two-particle production begins at

$$\omega>ck. \tag{G4}$$

The corresponding negative-frequency cuts follow by the usual reality relation. This immediately makes the causal origin nonanalytic: the Landau cut reaches $(\omega,k)=(0,0)$, so different approaches to the origin need not agree. In particular,

$$\lim_{\omega\to 0}\lim_{k\to 0}\Pi_{TT}^{R}(\omega,k)\neq\lim_{k\to 0}\lim_{\omega\to 0}\Pi_{TT}^{R}(\omega,k). \tag{G5}$$

Equation (G5) is why the Euclidean Kubo stiffness of Appendix C and the collisionless retarded origin are not interchangeable. The former is the equilibrium hydrostatic projection; the latter probes a different dynamical limit of the same completed matter response.

For fixed real k, the collisionless principal-sheet thermal kernel is most stably reconstructed from its exact real-axis discontinuity rather than by evaluating a nearly singular two-dimensional loop integral directly. Let $R(z,k)$ denote the dimensionless nonlocal thermal factor after the same local polynomial subtraction used to define the causal inverse response, and let R_I denote its principal-sheet reconstruction. Then

$$R_I(z,k)=\frac{2}{\pi}\int_0^{\infty}d\omega\,\frac{\omega\,\mathrm{Im}\,R(\omega+i0,k)}{\omega^2-z^2}. \tag{G6}$$

The Landau/scattering and two-particle-production intervals are integrated independently and then combined. The scalar, Weyl and gauge-vector pieces use the same physical TT weights derived in Appendix B. At the benchmark wave number, Eq. (G6) reproduces an independent direct imaginary-axis evaluation of the full collisionless Standard-Model thermal kernel to better than one part in 10^9. That agreement is important because the causal-sheet search is sensitive to whether a numerical singularity is a genuine isolated pole or merely a finite representation of a continuum.

A decaying resonance above the two-particle-production cut is not a zero of the physical principal sheet. It appears after analytic continuation across the two-particle-production cut onto the adjacent retarded sheet. For the positive-frequency two-particle-production cut, let $Disc_P R$ denote the discontinuity of R across that two-particle-production cut. The subscript P used below denotes this positive-frequency two-particle-production cut and its adjacent retarded sheet.

$$R_{II,P}(z,k) = R_I(z,k) + Disc_P R(z,k), \tag{G7}$$

If s denotes the argument of the logarithmic factor whose branch cut is actually crossed, the corresponding logarithm is continued as

$$\log_{II,P} s = \log s - 2\pi i \tag{G8}$$

only for the logarithmic structure whose branch cut has actually been crossed. The Landau sheet has its own discontinuity and must be continued separately. A global rule in which every logarithm is replaced by $\log - 2\pi i$ does not reproduce the retarded Riemann surface.

Let $F_{II,P}(u,y)$ denote the dimensionless zero-forming inverse-response eigenvalue on the positive-frequency two-particle-production sheet. At

$$y_0 = 0.758224684, \tag{G9}$$

the converged two-particle-production-sheet root is

$$u_P(y_0) = 1.353089335 - 0.093439111\, i. \tag{G10}$$

The root derivative is

$$\left|\partial_u F_{II,P}(u_P, y_0)\right| = 5.39112. \tag{G11}$$

Thus

$$F_{II,P}(u_P, y_0) = 0, \partial_u F_{II,P}(u_P, y_0) \neq 0. \tag{G12}$$

The resonance is therefore a simple pole of the compliance, not an exceptional point. Its damping ratio is

$$\frac{-\operatorname{Im} u_P}{\operatorname{Re} u_P} = 0.0691, \tag{G13}$$

so the oscillatory part dominates strongly in the ordinary state.

Tracking the same root as the real wave number is changed gives the numerical continuation summarized in Table G1.

Table G1. Numerical continuation of the simple underdamped resonance on the two-particle-production sheet with real wave number.

y	$u_P(y)$	$\lvert\partial_u F\rvert$
0.4	$0.9600 - 0.0629i$	≈ 5.92
0.7582	$1.35309 - 0.0934$	5.39
1.0	$1.58615 - 0.1056$	≈ 5.44
2.0	$2.48635 - 0.1177$	≈ 6.51
3.0	$3.38920 - 0.1055$	≈ 8.15
4.0	$4.31703 - 0.0897$	≈ 10.07
5.0	$5.26477 - 0.0756$	≈ 12.15

The branch remains timelike,

$$\mathrm{Re}\, u_P > y\,, \tag{G14}$$

through this scan, and the derivative moves away from zero rather than toward it. Extending the numerical continuation to much larger y preserves the same qualitative topology: the root remains underdamped, and its fractional damping eventually decreases.

The Landau sector supplies an independent numerical check. A finite quadrature of a branch cut necessarily replaces the continuum by a large set of rational poles. A naïve root search therefore finds many apparent zeros immediately below the Landau interval. Their fate under resolution refinement identifies them. One representative sequence is summarized in Table G2.

Table G2. Resolution test for apparent Landau-cut poles as the continuum quadrature is refined.

N_L	apparent root
180	$0.395596 - 1.1492 \times 10^{-}$
240	$0.396432 - 8.6301 \times 10^{-}$
360	$0.397270 - 5.7608 \times 10^{-}$
420	$0.397510 - 4.9396 \times 10^{-}$

Let N_L denote the number of quadrature nodes used to resolve the Landau interval. For this sequence,

$$N_L \lvert \mathrm{Im}\, u \rvert \approx constant\,, \tag{G15}$$

so

$$\lvert \mathrm{Im}\, u \rvert \propto \frac{1}{N_L}\,. \tag{G16}$$

The apparent poles therefore collapse onto the real cut as the continuum is resolved. A second representative sequence behaves the same way. The free collisionless Landau sector is consequently a continuum rather than a hidden set of stable long-lived modes.

Equations (G10)-(G16) also distinguish the physical retarded resonance from the analytic double zero of the leading one-log representation. The latter is a genuine mathematical feature, but it lives on the principal logarithmic sheet. Let x be the dimensionless invariant coordinate of that representation and let $A>0$ be its dimensionless matter-response coefficient. Define

$$F_{an}(x;A)=1+\frac{x\log x}{A}. \tag{G17}$$

Its nontrivial roots obey

$$x\log x=-A. \tag{G18}$$

Writing $w=\log x$, Eq. (G18) becomes $we^{w}=-A$. With the Lambert function defined by $W(q)e^{W(q)}=q$, the two real principal solutions use the branches W_0 and W_{-1}:

$$x_{\pm}=\exp\left[W_{0,-1}(-A)\right]. \tag{G19}$$

At

$$A_c=\frac{1}{e}, \tag{G20}$$

the two branches meet at

$$x_c=\frac{1}{e}. \tag{G21}$$

Direct differentiation gives

$$F_{an}'(x)=\frac{\log x+1}{A}, F_{an}''(x)=\frac{1}{Ax}, \tag{G22}$$

and hence

$$F_{an}(x_c)=0, F_{an}'(x_c)=0, F_{an}''(x_c)=e^2\neq 0. \tag{G23}$$

Thus the principal-sheet merger is an exact nondegenerate double zero of the leading analytic representation. The assembled leading interaction correction changes the coefficient that maps Standard-Model matter into A but leaves the Lambert-W branch topology itself intact. The leading thermal-memory calculation likewise yields a central dimensionless principal-sheet benchmark of

$$u_{an}=3.507691515625. \tag{G24}$$

Equation (G23), however, does not imply that the physical retarded resonance on the two-particle-production sheet must satisfy the same derivative condition. Crossing the two-particle-production cut adds its discontinuity and changes the analytic function whose zero is being followed. The explicit two-particle-production-sheet continuation establishes precisely this separation: the principal-sheet double zero does not continue into the same double zero on the retarded two-particle-production sheet. The latter contains the simple resonance of Eq. (G10).

There is a useful codimension reason not to expect a generic real-k exceptional point at a fixed physical state. The conditions

$$F(u,y)=0\,,\partial_u F(u,y)=0 \tag{G25}$$

are two complex equations, hence four real conditions. At fixed state and real wave number the available variables are

$$\mathrm{Re}\,u\,,\mathrm{Im}\,u\,,y\,, \tag{G26}$$

only three real coordinates. A generic real-k exceptional point therefore requires an additional physical state coordinate or a special symmetry/kinematic constraint. This codimension argument does not forbid an exceptional point, but it explains why one should not be expected merely by scanning real k in a fixed KMS state.

The first controlled interaction analysis can now be organized around how it might change this simple-root topology. The effects separate into thermal dispersion, quasiparticle widths, threshold motion, and the remaining finite frequency-dependent Ward-completed self-energy/vertex deformation. The first two have exact multiplicity statements.

For a common hard energy width $\Gamma\equiv\hbar\gamma_{hard}$, where γ_{hard} is the corresponding decay rate, suppose the dominant effect is the analytic translation

$$F_\Gamma(u,y)=F_0(u+i\Gamma/T\,,y). \tag{G27}$$

Then

$$u_\Gamma=u_0-i\Gamma/T\,,F_\Gamma{}'(u_\Gamma,y)=F_0{}'(u_0,y). \tag{G28}$$

A common width moves the pole deeper into the lower half plane but cannot make a simple root double.

Similarly, let $s\equiv u^2-y^2$ denote the dimensionless invariant of the unperturbed branch and let δm^2 denote a dimensionless reactive dispersion shift. If the thermal dispersion correction acts as the invariant translation

$$s=u^2-y^2\,,F_m(u,y)=F_0(s-\delta m^2)\,, \tag{G29}$$

then a simple zero $F_0(s_0)=0$, $F_0{}'(s_0)\neq 0$, is carried to

$$s_m=s_0+\delta m^2 \tag{G30}$$

with

$$\partial_u F_m=2u_m F_0{}'(s_0). \tag{G31}$$

For $u_m \neq 0$, a pure thermal-mass translation likewise cannot change the root multiplicity.

Threshold crossing has to be treated on the Riemann surface rather than by differentiating a square-root singularity in the original variable. Near a two-particle threshold u_{th}, introduce the uniformizing coordinate

$$w = \sqrt{u - u_{th}}. \tag{G32}$$

The completed inverse response is locally analytic in w,

$$F(w) = A + Bw + C\,w^2 + \cdots. \tag{G33}$$

A simple pole crossing the threshold satisfies

$$F(w_p) = 0\,,\, \partial_w F(w_p) \neq 0\,, \tag{G34}$$

and simply moves from one sheet to the adjacent one. An exceptional point at the threshold requires the additional independent condition

$$F(0) = 0\,,\, \partial_w F(0) = 0\,. \tag{G35}$$

Therefore

$$\text{pole reaches a two-particle-production threshold} \not\Rightarrow \text{exceptional point}\,. \tag{G36}$$

The angular momentum of the TT source further controls the strength of the threshold singularity. Let $\rho_{TT,s}$, $\rho_{TT,f}$, and $\rho_{TT,V}$ denote the scalar, fermion, and vector TT spectral densities near their respective two-particle thresholds, and let β denote the usual dimensionless relative-velocity parameter that vanishes at threshold. For a scalar, let s_{th} denote the value of the dimensionless invariant $s = u^2 - y^2$ at the two-particle threshold. Then

$$T_{xy} \sim p_x\, p_y\,,$$

so the threshold matrix element behaves as p^2. Including two-body phase space gives

$$\rho_{TT,s} \propto p^5 \propto (s - s_{th})^{5/2}. \tag{G37}$$

A fermion-antifermion two-particle state must carry at least one unit of orbital angular momentum to form total $J = 2$, so its TT threshold is at least P-wave,

$$\rho_{TT,f} \propto \beta^3 \text{ or softer}\,. \tag{G38}$$

Two spin-one particles can form $J = 2$ in an S-wave, so the vector threshold can begin as

$$\rho_{TT,V} \propto \beta\,. \tag{G39}$$

The non-Abelian gauge-boson thresholds are consequently the only nearby thresholds with the strongest square-root onset.

At the central running state and the benchmark momentum of Eq. (G9), the leading asymptotic thermal masses place the principal real parts of the gluon (g), weak-gauge-boson (W), and hypercharge-gauge-boson (B) two-particle-production thresholds at the dimensionless values

$$u_{th,g}^{(0)} \equiv \frac{\hbar\omega_{th,g}}{T} = 1.233752\,, u_{th,W}^{(0)} \equiv \frac{\hbar\omega_{th,W}}{T} = 1.225000\,, u_{th,B}^{(0)} \equiv \frac{\hbar\omega_{th,B}}{T} = 1.187933\,. \tag{G40}$$

All remain below the real part of the causal resonance,

$$\mathrm{Re}\, u_P = 1.353089\,. \tag{G41}$$

Finite quasiparticle widths move the nearest branch points into the lower half plane. Using the leading-log hard widths gives approximately

$$u_{th,g} = 1.233752 - 0.0814378\, i\,, \tag{G42}$$

and

$$u_{th,W} = 1.225000 - 0.0553086\, i\,. \tag{G43}$$

Their complex distances from the collisionless two-particle-production-sheet pole are

$$\left|u_P - u_{th,g}\right| = 0.119939\,, \left|u_P - u_{th,W}\right| = 0.133644\,. \tag{G44}$$

The pole is close enough that the exact finite interaction correction matters for its precision coordinate, but it is not coincident with either branch point.

Let F_0 denote the unperturbed two-particle-production-sheet inverse-response factor and let $u_0 \equiv u_P(y_0)$ be its simple root at the benchmark momentum. If δF denotes the interaction deformation of that factor, the known nonzero root derivative quantifies how far the root moves. To first order,

$$\delta u = -\frac{\delta F(u_0, y)}{F_0{}'(u_0, y)} + O(\delta F^2)\,. \tag{G45}$$

Using Eq. (G11), moving the pole to the dressed gluon branch point requires

$$\left|\delta F\right|_{gluon} \simeq 5.39112(0.119939) = 0.64661\,, \tag{G46}$$

while reaching the W branch point requires

$$\left|\delta F\right|_{W} \simeq 0.72049\,. \tag{G47}$$

These are order-one deformations of the normalized zero-forming inverse response. A simple estimate based only on the raw thermal mass scale is therefore misleading: the relevant question is how much the *completed inverse response* must change, not how large an individual quasiparticle self-energy is.

The hard Standard-Model TT interaction algebra provides a second scale check. After the common tensor normalization, define C_λ as the Higgs self-coupling weight, $C_{g(g)}$ as the gauge-

gauge weight, C_{y_t} as the top-Yukawa weight, and C_{s-g} and C_{f-g} as the scalar-gauge and fermion-gauge weights. At the central state they are approximately

$$C_{\lambda}=-0.008397\,,C_{g(g)}=0.899790\,,C_{y_t}=0.121182\,, \tag{G48}$$

$$C_{s-g}=0.246368\,,C_{f-g}=2.274023\,. \tag{G49}$$

Using $1/(4\pi)^2=0.00633257$ only as a loop-scale diagnostic gives a natural combined deformation scale of roughly

$$\epsilon_{2loop}^{diagnostic}\simeq 0.0225\,, \tag{G50}$$

before the dimensionless master-function ratios are inserted. Equation (G50) is not a strict error bar; an individual master may enhance or cancel another. It nevertheless shows that the 0.65-0.72 deformation required to force the pole onto the nearest gauge threshold would need an enhancement of order thirty over the natural two-loop scale.

The actual pole kinematics do not lie in a known enhancement region. Define the dimensionless thermal invariant at the benchmark pole by $K_P^2\equiv u_P^2-y_0^2$. Its magnitude is

$$\left|K_P^2\right|=1.272590052\,. \tag{G51}$$

Let g_3, g_2, and g_Y denote the dimensionless four-dimensional strong, weak, and hypercharge gauge couplings evaluated at the central thermal scale. Then

$$\frac{\left|K_P^2\right|}{g_3^2}=5.37\,,\frac{\left|K_P^2\right|}{g_2^2}=5.04\,,\frac{\left|K_P^2\right|}{g_Y^2}=5.58\,. \tag{G52}$$

The external hard timelike pole therefore lies outside the dimensionless regime $\left|K_P^2\right|\lesssim g^2$, equivalently the physical soft-invariant regime of order g^2T^2, in which ordinary fixed-order thermal power counting becomes delicate. It is also separated from the light cone by

$$\mathrm{Re}\,u_P-y_0=0.594865\,, \tag{G53}$$

about eleven central hard two-particle-production widths. The soft gauge-exchange region requires the usual hard-thermal-loop (HTL) matching, but there is no known soft, collinear, polarization-mixing or kinetic singularity at the pole capable of supplying the order-thirty enhancement needed to change the local root count.

These observations support a local topology statement. Because

$$F_0(u_0,y_0)=0\,,F_0{}'(u_0,y_0)\neq 0\,, \tag{G54}$$

the implicit-function theorem guarantees one continuously connected root for a sufficiently small analytic interaction deformation inside any neighborhood that contains no branch point. The nearest dressed branch point is about 0.12 away in the dimensionless u-plane. Let $\rho_u=0.05$ be the radius of a local disk around the free root; that disk contains no known singularity, and the free inverse-response variation across its boundary is approximately

$$\left|F_0'\right|\rho_u \simeq 5.39112(0.05) = 0.270. \tag{G55}$$

Changing the local root count therefore requires an interaction correction comparable to the free inverse response on the boundary of that neighborhood, not an ordinary small pole shift. At the first controlled interacting order no such enhancement mechanism appears.

The ordinary branch also remains underdamped under intentionally excessive sensitivity tests. The dimensionless collisionless damping at the benchmark point is $-\mathrm{Im}\, u_P = 0.093439$. Adding the central hard two-particle-production energy width $\Gamma / T \simeq 0.05404$ gives the total dimensionless damping coordinate

$$\hat{\gamma} \equiv -\mathrm{Im}\, u_P + \Gamma / T \simeq 0.147479. \tag{G56}$$

Even assigning an unrealistically large squared reactive-dispersion shift $\left|\delta M^2\right| / T^2 = 0.6722$ directly to the collective pole yields representative dimensionless reactive energies

$$e_+ \equiv \frac{E_+}{T} = 1.58210,\, e_- \equiv \frac{E_-}{T} = 1.07641, \tag{G57}$$

with damping ratios only

$$\frac{\hat{\gamma}}{e_+} = 0.0932,\, \frac{\hat{\gamma}}{e_-} = 0.1370. \tag{G58}$$

Replacing the central added width by a broad $0.1\,T$ envelope still leaves the ratios at only about 0.122 and 0.180. Thus even a deliberately overstated interaction sensitivity does not drive the ordinary branch toward the $\zeta = 1$ mechanical critical-damping limit.

The finite-k dispersion itself provides additional information without changing that conclusion. At small real y, the tracked collisionless two-particle-production-sheet branch behaves approximately as

$$u_P^2(y) \simeq 2.28\, y,\, y \ll 1. \tag{G59}$$

At large real y,

$$u_P^2(y) - y^2 \to 2.931 - 0.480\, i. \tag{G60}$$

Equation (G60) is the asymptote of the *collisionless* thermal kernel. In the interacting theory the cleaner decomposition is

$$u_P(y) = E_P(y) - i\,\gamma_P(y), \tag{G61}$$

with the reactive dispersion scale and damping rate treated separately. The collisionless high-y dimensionless damping implied by Eq. (G60) behaves approximately as

$$\gamma_{free}(y) \simeq \frac{0.240}{y}. \tag{G62}$$

while the hard Standard-Model collisional energy width contributes a dimensionless amount $\Gamma/T = O(0.05)$. Equating the two gives a crossover near

$$y_{cross} \simeq 4.4. \tag{G63}$$

Beyond several thermal momenta, collisions dominate the damping rate, but

$$\frac{\gamma_P}{E_P} \sim \frac{0.05}{y} \tag{G64}$$

continues to fall. Interactions therefore make the high-y branch more realistically damped without pushing it toward critical damping.

A formal reduced-dispersion continuation of Eq. (G60) suggests complex spatial branch points, but the complete thermal kernel cannot be continued to complex k by the substitution $k \to i\kappa_{sp}$, where κ_{sp} is a real screening wave-number magnitude, inside the real-k discontinuity formula. The two-particle-production parameterization rotates internal momentum contours through poles of the Bose and Fermi distributions. The correct complex-k continuation therefore requires contour deformation plus the associated thermal residues. For this reason a reduced square-root branch point is a useful guide to spatial screening/coherence, but it is not identified here as an established exceptional point of the full kernel.

The static finite-wave-number projection supplies a final independent consistency check. The scalar, Weyl and vector thermal functions can be evaluated at zero Matsubara frequency while retaining finite y. Their small-y slopes reproduce the Kubo coefficients of Appendix C. When the leading resummed soft contribution is included, the resulting static TT factor has two positive simple roots,

$$F_{stat}(y_i) = 0,\, F_{stat}{}'(y_i) \neq 0,\, i = 1, 2. \tag{G65}$$

If the soft ring contribution is removed, these roots disappear and are replaced by a shallow positive minimum. The roots are therefore diagnostics of thermal spatial dispersion rather than universal critical correlation lengths.

The causal and static results can now be placed beside the proposed strong-field boundary without conflating them. A genuine critical state $S_\star$ must satisfy the full non-Markovian retarded conditions

$$K^R_{TT}(z_\star, k_\star; S_\star) = 0,\, \partial_z K^R_{TT}(z_\star, k_\star; S_\star) = 0, \tag{G66}$$

together with

$$\partial_z^2 K^R_{TT}(z_\star, k_\star; S_\star) \neq 0. \tag{G67}$$

The weakly coupled KMS pole of Eq. (G10) does not satisfy Eq. (G66), and nothing in the controlled interaction analysis suggests that ordinary matter naturally evolves toward it.

If an extreme state satisfying Eqs. (G66)-(G67) exists, define its actual pole energy

$$E_\star = \hbar\,\Omega_\star \tag{G68}$$

and its microscopic correlation length by

$$\xi_\star = \frac{\hbar\, c}{E_\star}. \tag{G69}$$

For the same state, define the state-specific macroscopic compliance by $G_{comp}(S_\star) \equiv \hbar\, c^5 / \left[16\pi\, A_{2,E}(S_\star)\right]$. The asymptotic Einstein-equivalent infrared field then assigns the localized state a gravitational radius

$$r_g(E_\star; S_\star) = \frac{2\, G_{comp}(S_\star)\, E_\star}{c^4}. \tag{G70}$$

The OQG strong-field self-loading identification is

$$r_g(E_\star; S_\star) = \xi_\star. \tag{G71}$$

The constitutive compliance relation for that same state is therefore

$$G_{comp}(S_\star) = \frac{\hbar\, c^5}{16\pi\, A_{2,E}(S_\star)}, \tag{G72}$$

and Eqs. (G68)-(G72) give

$$\frac{A_{2,E}(S_\star)}{E_\star^2} = \frac{1}{8\pi}. \tag{G73}$$

Equation (G73) is therefore a proposed constitutive boundary for a *separate extreme critical state*. It is not a normalization condition on the ordinary KMS pole, not a value imposed on the principal-sheet Lambert benchmark, and not an auxiliary ultraviolet cutoff.

The resulting causal-topology conclusion can now be stated without the older ambiguities:

$$\begin{aligned}&\text{massless long-range branch} + \text{one simple underdamped causal TT resonance}\\ &+\text{two-particle-production and Landau/scattering continua}.\end{aligned} \tag{G74}$$

The principal-sheet Lambert-W merger remains an exact analytic benchmark of the leading representation. The finite-y static roots remain thermal spatial diagnostics. Critical damping, if realized by the full response at all, belongs to a separate strong-field state. The exact off-shell finite $O(g^2)$ displacement of the ordinary pole on the two-particle-production sheet remains valuable as a precision determination of E_P, γ_P, and the reactive dispersion scale, while the controlled interaction analysis establishes the local topology relevant here: the ordinary weak-coupling branch is a simple underdamped matter resonance rather than a naturally emerging exceptional point.

Appendix H. Infrared completion, cross-state continuity, and claim ledger

Appendix H gathers the macroscopic consequences of the two-point calculation and, just as importantly, keeps their logical status separate. The derivation has four parts: construct the fixed-measure WTDiff representative, recover the minimal Einstein-equivalent infrared equation, show how homogeneous vacuum shifts and compact changes of state behave, and finish with a claim ledger that distinguishes calculated results from representation choices, constitutive boundaries, external numerical inputs, and empirical requirements.

The division of labor is the guiding principle. The complete Standard-Model stress response supplies the microscopic causal kernel and the state-dependent TT stiffness; the inverse kernel supplies the common long-range compliance; fixed-measure WTDiff supplies the selected massless two-helicity infrared representation; and conservation plus the contracted Bianchi identity complete the minimal local macroscopic equation. Low-energy threshold matching changes which microscopic degrees of freedom carry the Hessian without requiring the long-range branch to fragment into body-specific gravitational couplings.

The fixed-measure/WTDiff representation used here follows the massless spin-two and vacuum-shift structure developed in the WTDiff/unimodular literature, including Alvarez, Blas, Garriga and Verdaguer; Alvarez and Vidal; Carballo-Rubio; and the matter-coupling analysis of Alvarez. The fixed-measure representation is introduced only after the matter-response inversion of Appendix A. Let $q_{\mu\nu}$ be a nonsingular representative of the operational tensor response, let $q \equiv det\, q_{\mu\nu}$, and let $\varpi(x)$ be a fixed nowhere-vanishing scalar density. In four dimensions define

$$\hat{g}_{\mu\nu} = \left(\frac{\varpi}{\sqrt{-q}} \right)^{1/2} q_{\mu\nu} . \tag{H1}$$

Because a local conformal rescaling with nonzero factor $\Omega(x)$, $q_{\mu\nu} \to \Omega^2 q_{\mu\nu}$, sends $\sqrt{-q} \to \Omega^4 \sqrt{-q}$, the two factors in Eq. (H1) cancel and

$$\hat{g}_{\mu\nu} \text{ is invariant under } q_{\mu\nu} \to \Omega^2 q_{\mu\nu} . \tag{H2}$$

Its determinant is fixed by construction,

$$\sqrt{-\hat{g}} = \varpi . \tag{H3}$$

Thus the volume element is not an independently varied local dynamical degree of freedom in the infrared representative. Linearizing this representation gives the WTDiff redundancy used in Sec. 3. The microscopic Standard Model is not thereby declared Weyl invariant: masses, spontaneous symmetry breaking, running couplings, the Higgs improvement, and the trace anomaly remain properties of the matter theory and of its completed Hessian. WTDiff characterizes the long-range geometrical representative of the physical TT branch.

Let $R_{\mu\nu}$ and R denote the Ricci tensor and scalar of $\hat{g}_{\mu\nu}$, let $T_{\mu\nu}$ be the macroscopic matter stress, let $T \equiv \hat{g}^{\mu\nu} T_{\mu\nu}$ be its trace, let c be the local limiting signal speed, and let G_{comp} denote the macroscopic gravitational compliance. At the lowest local derivative order the gauge-preserving fixed-measure completion gives the trace-free field equation

$$R_{\mu\nu} - \frac{1}{4}\hat{g}_{\mu\nu} R = \frac{8\pi G_{comp}}{c^4}\left(T_{\mu\nu} - \frac{1}{4}\hat{g}_{\mu\nu} T\right). \tag{H4}$$

Here the macroscopic compliance is the reciprocal of the matter-generated stiffness,

$$G_{comp} = \frac{\hbar c^5}{16\pi A_{2,E}}, \tag{H5}$$

with the normalization fixed by the same TT response convention used in Appendices A-C. Equation (H4) is therefore not used to determine $A_{2,E}$; it is the infrared geometrical expression of the already-calculated response coefficient.

Let ∇_μ denote the covariant derivative compatible with $\hat{g}_{\mu\nu}$. Taking the covariant divergence of Eq. (H4) supplies the integrability condition. On the settled infrared branch the complete matter tensor is conserved,

$$\nabla^\mu T_{\mu\nu} = 0, \tag{H6}$$

while the contracted Bianchi identity gives

$$\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu} R\right) = 0. \tag{H7}$$

Combining Eqs. (H4)-(H7) yields

$$\nabla_\nu\left(R + \frac{8\pi G_{comp}}{c^4} T\right) = 0, \tag{H8}$$

and hence, on each connected settled branch, introduces a spacetime integration datum Λ defined by

$$R + \frac{8\pi G_{comp}}{c^4} T = 4\Lambda. \tag{H9}$$

Define the Einstein tensor by $G_{\mu\nu} \equiv R_{\mu\nu} - 1/2\, \hat{g}_{\mu\nu} R$. Substituting Eq. (H9) back into Eq. (H4) gives

$$G_{\mu\nu} + \Lambda \hat{g}_{\mu\nu} = \frac{8\pi G_{comp}}{c^4} T_{\mu\nu}. \tag{H10}$$

The minimal local two-derivative WTDiff completion is therefore Einstein-equivalent on the infrared branch. This is a definite completion statement about the selected massless two-helicity representation of the matter-generated long-wavelength TT response. It is distinct from the stronger microscopic question of whether the complete connected TTT, $TTTT$, and higher Standard-Model stress hierarchy uniquely generates every nonlinear vertex of Eq. (H10).

A fixed measure also makes the status of the homogeneous vacuum zero unusually transparent. Let C denote an arbitrary homogeneous c-number energy-density offset. Consider the shift of the stress baseline

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + C\,\hat{g}_{\mu\nu}, T \rightarrow T + 4C. \tag{H11}$$

The trace-free combination on the right-hand side of Eq. (H4) is unchanged identically. In the Einstein-equivalent form, the same transformation is absorbed by the relabeling

$$\Lambda \rightarrow \Lambda + \frac{8\pi G_{comp}}{c^4} C. \tag{H12}$$

The same result appears directly at the effective-action level. Let $\Delta\Gamma_C$ denote the change in the effective action produced by this homogeneous offset. Then

$$\Delta\Gamma_C = C\int d^4x\, \varpi(x), \tag{H13}$$

and therefore

$$\delta_{\hat{g}} \Delta\Gamma_C = 0, \tag{H14}$$

because the measure is fixed. The connected microscopic statement is consistent with Eq. (H14): a c-number offset does not alter

$$\delta T_{\mu\nu} = T_{\mu\nu} - \langle T_{\mu\nu} \rangle, \tag{H15}$$

and hence does not alter the connected retarded commutator, the physical TT spectral density, or the Kubo slope.

Equations (H11)-(H15) isolate the precise vacuum-energy result. An arbitrary homogeneous metric-proportional stress zero does not become a local connected gravitational source and does not require cancellation against a separately chosen local bare term. This statement does not remove vacuum physics. Vacuum polarization, Casimir and boundary stresses, phase-transition differences, condensates, trace-anomaly terms, curvature-dependent effective interactions, and stress fluctuations change connected state information and remain in the completed response. The result is blindness to an arbitrary *zero*, not blindness to physical vacuum structure.

The same distinction between an asymptotic branch and a localized state change controls the temperature problem. For a family of homogeneous high-state Kubo-Martin-Schwinger (KMS) branches, let T denote the equilibrium temperature, k_B Boltzmann's constant, and $a_T(T)$ the dimensionless high-state stiffness coefficient. Write

$$A_{2,E}(T) = a_T(T)(k_B T)^2. \tag{H16}$$

Using Eq. (H5),

$$G_{comp}(T) = \frac{\hbar c^5}{16\pi a_T(T)(k_B T)^2}. \tag{H17}$$

For two homogeneous branches T_1 and T_2,

$$\frac{G_{comp}(T_1)}{G_{comp}(T_2)} = \frac{a_T(T_2)\,T_2^2}{a_T(T_1)\,T_1^2}. \tag{H18}$$

Equation (H18) is a comparison law between homogeneous equilibrium branches of the high-state theory. It is not a prediction that a finite laboratory specimen acquires $G_{comp} \propto T_{lab}^{-2}$ when its thermometer reading is changed.

The present paper does not calculate the broken-phase electroweak, quantum-chromodynamic, nuclear, atomic, and condensed-matter matching coefficients explicitly. Let T_{lab} denote laboratory thermometer temperature, let $A_{2,E}^{GS}$ denote the zero-temperature coefficient that would result from a completed matched low-energy theory, and let $\delta A_{2,E}^{low}(T_{lab})$ denote the contribution of the actual low-energy thermal excitations. The generic matched form required by a continuous low-energy description is

$$A_{2,E}^{body}(T_{lab}) = A_{2,E}^{GS} + \delta A_{2,E}^{low}(T_{lab}),\, \delta A_{2,E}^{low} \to 0\,(T_{lab} \to 0). \tag{H19}$$

The coefficient $A_{2,E}^{GS}$ is not calculated numerically in the present high-state analysis; a nonzero low-temperature intercept is required if the observed common long-range gravitational branch is to persist in this continuation. Structurally, the active thermal degrees of freedom change through electroweak symmetry breaking, quantum-chromodynamic confinement, nuclear and atomic binding, and condensed-matter formation. In an explicit effective-field-theory matching, heavy thermal populations would be integrated out while their virtual, binding, and ground-state effects are encoded in the lower-energy coefficients. At laboratory temperatures the thermally active modes are therefore those of the material's low-energy theory rather than the unbroken high-temperature Standard-Model plasma.

For two finite bodies embedded in one common asymptotic branch, let K_∞ denote the static inverse response of the ambient branch and let ΔK_i denote the localized operator modification produced by body i. Then

$$K_{stat} = K_\infty + \Delta K_1 + \Delta K_2, \tag{H20}$$

where each ΔK_i has compact spatial support. Let G denote the exact static Green operator, defined as the inverse of K_{stat}:

$$G = K_{stat}^{-1}. \tag{H21}$$

The resolvent identity gives

$$G = G_\infty - G_\infty(\Delta K_1 + \Delta K_2)\,G, \tag{H22}$$

with

$$G_\infty = K_\infty^{-1}.$$

Let $A_{2,E}^{\infty}$ denote the TT stiffness of the common ambient branch and let ∇^2 be the spatial Laplacian. At long wavelength the ambient operator has the form

$$K_{\infty}=A_{2,E}^{\infty}\left(-\nabla^2\right)+O\left(\nabla^4\right), \tag{H23}$$

Let r denote the spatial separation between source and observation points. The ambient Green function therefore has

$$G_{\infty}(r)=\frac{1}{4\pi A_{2,E}^{\infty}r}+\text{subleading terms}, \tag{H24}$$

up to the tensor projector and the fixed source normalization. Because the insertions in Eq. (H20) are localized, they modify scattering amplitudes, short-distance structure, induced multipoles, and finite-size form factors. They do not replace the asymptotic operator K_{∞} outside the bodies. A second independent $1/r$ pole would require an additional asymptotic massless degree of freedom or a state modification extending to infinity, neither of which is generated by simply heating, cooling, or changing the phase of a compact sample.

The same conclusion is obtained from the conserved zero-momentum form factor of a complete composite body. At zero momentum transfer, the conserved stress couples to total energy. Let $M_i(0)$ be the reference mass of body i and let $\Delta U_i(T_{lab})$ be its change in internal energy. A change of internal state therefore changes the mass by the ordinary amount

$$M_i(T_{lab})=M_i(0)+\frac{\Delta U_i(T_{lab})}{c^2}, \tag{H25}$$

while changing finite-momentum structure and higher multipoles. It does not assign a second independent long-range gravitational charge to the body. Let G_{∞} denote the common asymptotic Newtonian coupling of the ambient branch. Consequently the leading two-body potential is

$$V_{12}(r)=-G_{\infty}\frac{M_1 M_2}{r}+O\left(r^{-3}\right), \tag{H26}$$

and

$$G_{12}^{monopole}=G_{\infty}. \tag{H27}$$

This is the reciprocal cross-state result used in Sec. 9. The common Green function supplies the same leading monopole residue to both bodies, while their internal states remain free to alter finite-size response.

The empirical constraints on this common infrared branch can be written compactly. Let g denote the common reference gravitational acceleration, let g_a be the acceleration of composition a, and let β_a be its small fractional deviation. Parameterize

$$g_a=\left(1+\beta_a\right)g, \tag{H28}$$

For two compositions a and b, define the Eötvös parameter η_{ab} by its leading small-deviation form

$$\eta_{ab} \simeq \left|\beta_a - \beta_b\right| \tag{H29}$$

The MICROSCOPE result consequently requires the long-range projection to suppress any such material dependence to approximately the 10^{-15} level in its tested regime. For the radiative branch, let m_{TT}^{IR}, v_{TT}^{IR}, and $N_{helicity}$ denote the infrared tensor mass, propagation speed, and number of propagating tensor helicities. Consistency with the observed long-range tensor sector requires

$$m_{TT}^{IR} = 0\,, v_{TT}^{IR} = c\,, N_{helicity} = 2\,, \tag{H30}$$

Millimetre-scale optical-clock redshift provides an additional operational test that matter-built clocks follow the same settled comparison field.

The infrared derivation and cross-state analysis are most useful when their claim boundaries are stated as carefully as their equations. The following ledger summarizes the claim status of the two-point program. “Derived/calculated” means that the stated result is obtained within the approximations and conventions explicitly developed in this manuscript. “Infrared completion” means a definite property of the selected minimal WTDiff representation rather than a result already generated from all higher connected Standard-Model kernels. “Constitutive boundary” denotes an OQG physical identification that is not imposed on the ordinary weak-coupling KMS state. “External numerical input” identifies information that has been reduced to a well-defined observable but has not yet been numerically supplied. The resulting status map is summarized in Table H1.

Table H1. Claim-status ledger for the completed OQG two-point program.

Item	Current status / claim boundary
Complete Standard-Model matter source	**Retained microscopic source.** Yang-Mills, Dirac/Yukawa, Higgs, and interaction stresses enter one completed conserved response.
CTP normalization	**Derived and fixed.** The symmetric source convention gives the one-half retarded-response normalization and the Ward contacts remain in the same Hessian.
Matter-first constitutive inversion	**Derived as response-variable inversion.** $D^R = \left(\Pi^R\right)^{-1}$ changes the causal orientation of the constitutive law; it does not create a new Hilbert-space degree of freedom.
Physical free TT spectral weight	**Derived.** Scalar:Weyl:vector $= 1:3:12$, giving $N_T = 283$ for the minimal Standard Model.
Static Kubo normalization	**Derived.** $A_{2,E} = \kappa / 4$ in the locked OQG source convention.

Item	Current status / claim boundary
Higgs curvature improvement	**Resolved by channel separation.** It is TT-null in the resolved propagating spectral sector but contributes to the static metric-variational Hessian through the local curvature contact. No special TT-closure value of ξ_H is inferred.
$O\left(g^2\right)$ perturbative static matching	**Closed analytically.** Hard, electric/Higgs, Ward/contact, two-scale local-R, and finite electric matching contain no additional unknown perturbative gravitational coefficient.
Magnetostatic finite constants	**External numerical input.** The remaining c_3^{NP} and c_2^{NP} are finite pure-3D-Yang-Mills observables; Appendix F defines their flat-lattice extraction and continuum conversion.
Ordinary finite-$\left(\omega, k\right)$ causal topology	**Calculated at the controlling order.** One simple underdamped resonance on the two-particle-production sheet plus two-particle-production and Landau/scattering continua; the first controlled interaction topology tests do not generate a nearby exceptional point.
Principal-sheet Lambert-W double zero	**Exact analytic benchmark of the leading representation.** It is not identified with the physical retarded resonance on the two-particle-production sheet.
Critical damping of ordinary weak-coupling matter	**Not the ordinary branch.** The calculated KMS resonance is underdamped; critical damping is reserved for a separate possible strong-field state.
Strong-field $1/\left(8\pi\right)$ relation	**Constitutive boundary.** Applied only to a hypothetical state whose full retarded kernel actually satisfies the non-Markovian double-zero condition. It is not an ordinary-state normalization or cutoff.
Fixed-measure/WTDiff tensor	**Infrared representation.** Introduced after matter-response reduction; it is not a claim of microscopic Weyl invariance of the Standard Model.
Einstein-equivalent macroscopic equation	**Minimal infrared completion result.** The local two-derivative WTDiff completion plus conservation/Bianchi gives Eq. (H10). Direct derivation of all nonlinear vertices from $TTT, TTTT, \ldots$ is a separate higher-point question.
Homogeneous vacuum-energy zero	**Locally source-blind.** A c-number metric-proportional shift leaves connected response unchanged and is absorbed into the integration datum in the fixed-measure completion. Physical vacuum polarization, boundaries, phase transitions, condensates, anomaly terms, and fluctuations remain.
High-state T^{-2} compliance comparison	**Homogeneous-branch law only.** Equation (H18) is not a laboratory thermometer law for a finite specimen.

Item	Current status / claim boundary
Broken-phase / low-temperature threshold coefficients	**Structural EFT continuation; coefficients uncalculated here.** Equation (H19) gives the generic matched form, but the electroweak-to-QCD-to-nuclear/atomic-to-condensed-matter coefficient matching is not performed in this paper and $A_{2,E}^{GS}$ is not numerically derived.
Finite-body private Newton constant	**Excluded at leading monopole order within the cross-state construction.** Compact state changes modify finite-size response but share the common asymptotic $1/r$ pole, Eq. (H27).
Weinberg-Witten relation	**Bounded structural statement.** A direct naive mapping to a conventional local fixed-background composite-graviton target is not established. The present construction does not claim unconditional theorem-level evasion solely from nonlocality or Legendre inversion.
Empirical infrared universality	**Required by observation.** The common branch must satisfy redshift, equivalence-principle, massless two-helicity, and luminal-propagation constraints in their tested regimes.

This ledger is deliberately a status map rather than a chronological history of the research. It prevents quantities belonging to different physical regimes from being recombined after they have been separated by the calculation. In particular, the static Euclidean stiffness is not the retarded origin limit; the ordinary KMS pole on the two-particle-production sheet is not the principal-sheet Lambert merger; neither is the proposed strong-field critical state; the Higgs improvement is not extra resolved TT spectral weight; and the non-Abelian magnetostatic constants are not ordinary electromagnetic-field control parameters.

Appendix H therefore closes the proof layer associated with the infrared and cross-state portions of the manuscript. The completed two-point chain can be read without introducing an additional normalization convention:

$$\begin{gathered}\text{complete Standard-Model stress response} \to A_{2,E}, D^R \\ \to \text{common massless TT compliance} \to \text{fixed-measure/WTDiff infrared} \\ \to \text{Einstein-equivalent macroscopic dynamics}.\end{gathered} \tag{H31}$$

The same chain keeps an arbitrary homogeneous stress zero out of the local connected source while preserving physical vacuum response, and it keeps compact changes of material state inside one common asymptotic gravitational branch. These are the two infrared consistency results required before the body of the paper turns to the scope of the completed two-point program.

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